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On the theory of fixed points for some classes of mappings

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Analisi funzionale. — *On the theory of fixed points for some classes of mappings.* Nota di VASILE I. ISTRĂŢESCU e ANA ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema di surgettività per l'operatore I–A–B ove A è una contrazione e B una α–contrazione. Usando poi un lemma di M. Martelli si dà una nuova dimostrazione di un risultato precedente degli Autori.

1. Let X be a Banach space and B be a map defined on X. Following Granas, B is quasi-bounded if the number defined by

$$|B| = \limsup_{\|x\| \to \infty} \frac{\|Bx\|}{\|x\|}$$

is finite and $|B|$ is called the quasi-norm of B. It is clear that every operator B which is asymptotic to a linear operator in the sense of Dubrowski [2] is quasi-bounded.

Our purpose in the present Note is to give some extensions of results proved by Nashed and Wong using the notion of α–contraction. Also using a recent Lemma of Martelli we give a new proof of our earlier extension of Schauder fixed point theorem.

First we can prove the following, generalizing theorem 1 of [6].

THEOREM 1. *Let* A *be strict contraction on* X *and* B *an α–contraction and quasi-bounded with the quasi-norm*

1) $|B| < 1 - r$;

2) *the α–constant of* B *is less than* $(1 - r)$ *where r is the contraction constant of* A.

Then R (I–A–B) = X.

Proof. Let z be any fixed element of X and if y is in X we define

$$\tilde{A}_y x = Ax + Bz + y$$

and clear $\tilde{A}_y$ is a strict contraction and thus for z we may associate the fixed point of $\tilde{A}_y$, Lz which has the property

$$Lz = \tilde{A}Lz = ALz + Bz + y .$$

For any $u, v \in X$ we have

$$\|Lu - Lv\| = \|ALu - ALv + Bu - Bv\| \leq r\|Lu - Lv\| + \|Bu - Bv\|$$

(*) Nella seduta del 16 giugno 1972.

which gives that

$$(*) \qquad \|Lu - Lv\| \leq \frac{1}{1-r} \|Bu - Bv\| .$$

From this it is clear that L is α–contraction.

Let $S_N(y) = \{x, \|x-y\| \leq N\}$. We claim that there exists N such that $LS_N \subset S_N$. If this is not so then for every n, there exists $u_n \in S_n$ such that

$$\|Lu_n - y\| > n .$$

If $\{u_n\}$ is bounded then $\{Lu_n\}$ is a relatively compact set. Indeed $\{Lu_n\}$ is a bounded set. If $\alpha(Lu_n) > 0$ we have that

$$\alpha(Lu_n) < \frac{1}{1-r} \alpha(Bu_n) < \frac{k}{1-r} \alpha(\{u_n\})$$

$$A = \cup L^m(\{u_n\})$$

$$A = \bigcup_{i=1,2} L^i(\{u_n\}) .$$

From the estimate (*) it follows, because B is continuous, that L is also continuous and that if $\{u_n\}$ is bounded then $\{Lu_n\}$ is bounded which is a contradiction. Thus $\|u_n\| \to \infty$.

The estimates of Nashed and Wong work also in this case and we obtain the desired assertion. The application of Darbo–Sadovski theorem gives the existence of a fixed point for L and the conclusion of the theorem.

Remark. We can use the Hausdorff measure of noncompactness to obtain a new version of the above theorem. Here we have used the Kuratowski number which is an old measure of noncompactness (Kuratowski, 1929).

We remark that the technique in Nashed and Wong and that above give the following extension of Theorem 1.

THEOREM 2. *Let* A *be a bounded linear operator such that for some* $p > 1$, A^p *is a strict contraction and* B *be an* α*–contraction, quasi-bounded with the quasi-norm*

1) $|B| < (1-r)\left(\sum_{i=0}^{p-1} \|A\|^i\right)$;

2) *the constant of* α*–contr.* B *is less than* $(1-r)\Big/\sum_{i=0}^{p-1}\|A\|^i$ *where* r *is the constant contraction of* A^p, *then* $R(I-A-B) = X$.

Remark. It is clear that some of the above results are valid for mappings A not contractions, but "locally contraction" i.e., for each $x \in X$ there exists $k(x) \in [0, 1)$ such that for all $y \in X$

$$\|Ax - Ay\| \leq k(x)\|x - y\| .$$

Remark. Another generalization of the Theorem 3 of the paper [6] may be obtained considering that A is a mapping satisfying the property given in the Theorem 1 of [6], and we omit this. Also, if B is strictly semicontractive mapping with the constant k of the some type we can prove the Theorem 3 of [6]. In this case it is known that every strictly semicontractive mapping is α–contraction.

2. In [4] using the Kuratowski number we have proved a generalization of the theorem of Schauder. In [5] M. Martelli gives a simple lemma from which a new proof for the theorem of Sadovski and Furi–Vignoli follows. In this section we show that the lemma of Martelli may be used to prove our theorem.

For A, a bounded subset of a Banach space X, we denote by $\alpha(A)$, the Kuratowski number of A, the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε. We denote by $\chi(A)$, the Hausdorff numbers of A, the infimum of all $\varepsilon > 0$ for which A has a finite ε–net.

DEFINITION 2.1. *Let* Q *be a nonempty subset of* X *and* $T : Q \to X$ *a continuous mapping.* T *is called locally power densifying (Kuratowski) if for each bounded subset* A *of* Q *there exists an integer* $n = n(A)$ *such that*

$$\alpha(T^n A) < \alpha(A)$$

and is called locally power densifying (Hausdorff) if for some integer $n_1 = n(A)$ *have*

$$\chi(T^{n_1} A) < \chi(A).$$

In [4] the extension of Schauder theorem was proved for a subclass of mappings defined in Definition 2.1, representing a generalization of the class of α–contractions of Darbo [1].

THEOREM 2.2. [4]. *If* T *is a mapping which is locally power densifying (Hausdorff) or locally power densifying (Kuratowski) and* Q *is a nonempty closed bounded convex set of a Banach space* X. *Then* T *has at least one fixed point in* Q.

Proof. Let x_0 be arbitrary point in Q and consider the set

$$A = \{T^n x_0\}_0^\infty$$

which is clear invariant for T. Since T is locally power densifying (Kuratowski or Hausdorff) we obtain easily that $\alpha(A) = 0$, since if $N = n(A)$ for instance,

$$A = \{x_0, \cdots, T^{N-1} x_0\} \cup \{T^N A\}$$

and thus, if $\alpha(A) > 0$

$$\alpha(A) = \alpha(T^N A) < \alpha(A)$$

which is a contradiction. Since $\alpha(A) = \alpha(\bar{A})$ (the bar denotes the closure) we have that $\bar{A}$ is a compact set. But $\bar{A}$ is invariant under T. Let M be the subset of $\bar{A}$ whose existence is ensured by the lemma of Martelli [5] and put as in [5],

$$\mathfrak{F} = \{ B , B \subset Q , M \subset B , B \text{ is closed convex and invariant under } T \} .$$

Define $F = \cap \{ B , B \in \mathfrak{F} \}$ and it is clear closed, convex and invariant under T. Also $F = \overline{co}(TF)$ where $\overline{co}(\cdot)$ denotes the closure of $(\cdot)$. If $N = n(A)$ then $T^N A \subset \overline{co}(F)$ and thus F is compact. This gives the existence of the fixed point for T and the theorem is proved.

Remark. Among the operators, which are in the class defined above is the following [7] $T : Q \rightarrow Q$ and for each $x \in Q$ there exists $n = n(x)$ such that for every $y \in Q$

$$\| T^n x - T^n y \| \leq k \| x - y \|$$

where $k \in [0 , 1)$ and is independent of x and y.

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