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Geometria. — *Algebraic Contractions and Complete Intersections.* Nota di LUCIAN BĂDESCU e MIHNEA MOROIANU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studiano dilatazioni e contrazioni di varietà algebriche, con particolare riguardo al caso in cui queste siano delle intersezioni complete.

In the following K will denote a commutative, algebraically closed field of arbitrary characteristic. We shall consider only irreducible algebraic varieties over K.

Let X′ be a projective algebraic variety embedded in the projective space P_n, such that it is a complete intersection in P_n, i.e. its homogeneous ideal $I(X')$ is generated by $n - \dim X'$ independent elements (which are even homogeneous forms) of $K[T_0, \cdots, T_n]$. $O_{X'}(1)$ stands for the inverse image of $O_{P_n}(1)$ by this embedding, while $S(X')$ represents the homogeneous coordinates ring of X′, i.e. $K[T_0, \cdots, T_n]/I(X')$.

We shall need the following results:

(A) [FAC, § 78, Proposition 5]. – *If* X′ *is a complete intersection in* P_n *defined by the equations* $f_i = 0$, $i = 1, \cdots, n$–dim X′, *of degrees* m_i, *then*:

a) *The canonical homomorphism of graded algebras*:

$$\alpha : S(X') \longrightarrow \bigoplus_{s \in Z} H^0(X', O_{X'}(s))$$

is an isomorphism;

b) $H^q(X', O_{X'}(s)) = 0$ *for every s and* $0 < q < \dim X'$;

c) *The vector space* $H^d(X', O_{X'}(s))$ (*with* $d = \dim X'$) *is isomorphic with the dual of* $H^0(X', O_{X'}(N - s))$, *where* $N = \sum_{i=1}^{n-d} m_i - n - 1$.

(B) [SGA–1962, Corollaire (3.7)]. – *Let* X′ *be an algebraic variety of dimension* ≥ 3, *which is embedded in* P_n *as a complete intersection. Then Pic* X′ (*the group equivalence classes of isomorphic invertible sheaves*) *is cyclic and generated by the class of* $O_{X'}(1)$.

Remarks. – 1) If dim X′ = 2, and if X′ is a complete intersection in P_n, Pic X′ is finitely generated by a result of Hartshorne [6], but not in general cyclic. For instance $X' = P_1 \times P_1$ is the quadric $T_0 T_1 - T_2 T_3 = 0$ in P_3 and Pic $X' = Z \times Z$, hence (B) does not hold in this case.

2) If dim X′ = 1, e.g. if X′ is an elliptic curve, Pic X′ has not a finite number of generators, though it is a plane curve.

(*) Nella seduta del 16 giugno 1972.

Let Y be an algebraic variety, $y \in Y$ a (closed) point, $\varphi : X \to Y$ the blowing-up of Y at y, $X' = \varphi^{-1}(y)$ the exceptional locus, such that we have the cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad i \quad} & X \\ {\scriptstyle \varphi'} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \operatorname{Spec} K = y & \xrightarrow{\quad j \quad} & Y \end{array}$$

j and i being the corresponding closed immersions given by the ideals J and $I = O_X(1)$ respectively. We assume that the following conditions are fulfilled:

a) There exists an embedding of X' in a projective space P_n, such that X' is complete intersection in P_n;

b) The conormal sheaf $i^* I = I/I^2$ of X' in X is isomorphic to a strictly positive tensor power of $O_{X'}(1)$, i.e. $i^* I = O_{X'}(s)$ with $s > 0$.

Remark. – If $\dim Y \geq 4$, it follows from (B) that *b*) is a consequence of *a*) because $i^* I$ is ample on X'.

PROPOSITION 1. *Let* Y *be an algebraic variety,* $y \in Y$ *a normal point,* $\varphi : X \to Y$ *the blowing-up of* Y *at* y, $X' = \varphi^{-1}(y)$. *Assume that the conditions a*) *and b*) *are fulfilled. Then*:

i) $R^q \varphi_* (I^n) = 0$ *if* $q > 0$, $q \neq \dim Y - 1$, *and* $n \geq 0$;

ii) *if further,* $R^1 \varphi_* (I^n) = 0$ *for* $n \geq 1$, *then the canonical homomorphism*

$$\alpha : \bigoplus_{n \geq 0} J^n \longrightarrow \bigoplus_{n \geq 0} \varphi_* I^n$$

is an isomorphism.

Proof. – The case $q > \dim Y - 1$ is a consequence of EGA III (4.4.2), therefore we can assume that $0 < q < \dim Y - 1$. Since $I = O_X(1)$ is φ–very ample and φ is a proper morphism, we have $R^q \varphi_* (I^n) = 0$ for $n \gg 0$. On the other hand, from *b*) we deduce the exact sequence:

$$(1) \qquad 0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow i_* O_{X'}(sn) \longrightarrow 0\,,$$

which gives rise to the exact sequence:

$$(2) \qquad R^q \varphi_* (I^{n+1}) \longrightarrow R^q \varphi_* (I^n) \longrightarrow R^q \varphi_* (i_* O_{X'}(sn))\,.$$

But $R^q \varphi_* (i_* O_{X'}(sn)) = j_* R^q \varphi'_* O_{X'}(sn) = H^q(X', O_{X'}(sn)) = 0$ [by (A)], consequently the canonical homomorphism:

$$R^q \varphi_* (I^{n+1}) \longrightarrow R^q \varphi_* (I^n)$$

is surjective and the required property follows by descending induction on n.

In order to prove ii) we remark that from the normality of y on Y it follows $\varphi_*(O_X) = O_Y$. In fact, consider Stein's factorisation

$$(3)\qquad \begin{array}{ccc} X & \xrightarrow{\varphi_2} & Y_1 = \operatorname{Spec} \varphi_*(O_X) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi_1} & \\ & Y & \end{array}$$

where $\varphi_*(O_X)$ is a coherent O_Y–algebra and, therefore, φ_1 is a finite morphism [EGA III (3.2.1) and II (6.1.3)]; we conclude remarking that φ_1, in view of Zariski Main's Theorem, is an isomorphism, since it establishes an isomorphism between $Y_1 - \varphi_1^{-1}(y)$ and $Y - y$.

Next we prove that $\varphi_* I = J$; this follows from the commutative diagram

$$(4)\qquad \begin{array}{ccc} O_Y & \longrightarrow & j_* O_{Y'} \\ \wr\downarrow & & \downarrow\wr \\ \varphi_*(O_X) & \longrightarrow & \varphi_* i_* O_{X'} \end{array}$$

in which the first vertical arrow is the isomorphism established above, $Y' = y$, the second one is the isomorphism which follows from $\varphi_* i_* O_{X'} = i_* \varphi'_* O_{X'} = = j_* O_{Y'}$ and the kernels of the horizontal arrows are J and $\varphi_*(I)$ respectively.

It remains to be proved that α_n is an isomorphism for $n \geq 2$. From the commutative diagram

$$(5)\qquad \begin{array}{ccc} J^n & \xrightarrow{\alpha_n} & \varphi_*(I^n) \\ \downarrow & & \downarrow \\ O_Y & \underset{\sim}{\xrightarrow{\alpha_0}} & \varphi_*(O_X) \end{array}$$

we deduce that α_n is injective. From the exact sequence (1), and since $R^1 \varphi_* I^{n+1} = 0$, one can obtain the exact sequence:

$$(6)\qquad 0 \longrightarrow \varphi_* I^{n+1} \longrightarrow \varphi_* I^n \longrightarrow \varphi_* i_* O_{X'}(ns) \longrightarrow 0 .$$

Therefore $\varphi_* I^n/\varphi_* I^{n+1} \cong \varphi_* i_* O_{X'}(ns) = j_* \varphi'_* O_{X'}(ns)$; we get the commutative diagram

$$(7)\qquad \begin{array}{ccc} J^n/J^{n+1} & \xrightarrow{\bar{\alpha}_n} & \varphi_*(I^n)/\varphi_*(I^{n+1}) \\ & {\scriptstyle\alpha'_n}\searrow \quad \swarrow\wr & \\ & j_* \varphi'_* O_{X'}(ns) & \end{array}$$

in which α'_n is the canonical homomorphism [EGA II (3.3.2)] and all the involved sheaves are concentrated in y. But $\bar{\alpha}_1$ is a surjection, since α_1 is an isomorphism; hence α'_n is a surjection too, and therefore α_n is surjective for every $n \geq 1$ [because of (A) which implies that the graded algebra $\bigoplus_{n\geq 0} \varphi'_*(O_{X'}(ns))$ is generated by its component of degree one]. In this way, $\bar{\alpha}_n$ is an isomorphism and then α_n is an isomorphism too, using the fact that this is true for $n \gg 0$ [EGA III (2.3.1)].

It is useful to remark that the additional hypothesis $R^1 \varphi_* I^n = 0$ is always fulfilled if $\dim Y \geq 3$ [by (i)].

COROLLARY. – *In the above conditions we have:*

(i) *The canonical homomorphism*

$$\alpha' : \bigoplus_{n\geq 0} J^n/J^{n+1} \longrightarrow \bigoplus_{n\geq 0} \varphi'_*(O_{X'}(ns))$$

is an isomorphism;

(ii) *There is a canonical isomorphism*

$$\alpha'' : \bigoplus_{n\geq 0} H^0(X', J^n/J^{n+1}) \longrightarrow S(X')^{(s)},$$

where $S(X')$ *is the graded* K*–algebra such that* $(S(X')^{(s)})_n = S(X')_{ns}$ *[see EGA II].*

Proof. – (i) follows considering again the commutative diagram (7) in which $\bar{\alpha}_n$ is this time an isomorphism, since this is true for α_n; (ii) follows from (i) and from (A).

Remark. – Using the same notations as in Proposition 1, if O_y is the local ring of y on Y and m_y its maximal ideal, the above corollary points out the existence of a canonical isomorphism

$$\alpha'' : \bigoplus_{n\geq 0} m_y^n/m_y^{n+1} \longrightarrow S(X')^{(s)}.$$

In particular, the dimension of Zariski's tangent space of Y at y is exactly $\dim_K S(X')_s$ [the dimension of the s–th component of the graded algebra $S(X')$].

If $X' = P_{r-1}$, then $S(X') = K[T_1, \cdots, T_r]$ and $\dim m_y^n/m_y^{n+1} = \binom{ns+r-1}{r-1}$, while the multiplicity of the local ring O_y is s^{r-1} (see [3]).

On the other hand, if X′ satisfies *a*) and *b*) with $s = 1$, then $gr_{O_y}(m_y)$ is isomorphic to $S(X')$; in other words, the dimension of Zariski's tangent space of Y at y and the multiplicity of O_y respectively coincide with the dimension of Zariski's tangent space of the affine cone defined by $S(X')$ at its vertex and the multiplicity of the local ring corresponding to the vertex of this cone.

Next we make some remarks concerning the blowing-up of an algebraic surface Y at a normal point, y, whose exceptional locus is an elliptic curve. In this case the condition *a*) fulfilled because X′ is a plane curve of degree 3 and the condition *b*) can be stated in the form:

b′) $\deg i(I) = 3s$ with s an integer, $s > 0$.

In fact it is sufficient to observe that $\deg O_{X'}(1) = 3$ on an elliptic curve; conversely, every invertible sheaf of degree 3 is very ample and gives rise to an imbedding of X′ in the projective plane (straightforward consequence of Riemann–Roch's theorem).

PROPOSITION 2. – *Let* Y *be an algebraic surface,* $y \in Y$ *a normal point,* $\varphi : X \to Y$ *the blowing-up of* Y *at* y. *Assume that the exceptional locus is an elliptic curve and that* $\deg i^* I = 3s$ *with* s *a strictly positive integer. Then the canonical homomorphism*

$$\alpha : \bigoplus_{n \geq 0} J^n \longrightarrow \bigoplus_{n \geq 0} \varphi_*(I^n)$$

is an isomorphism, and $R^q \varphi_*(I^n) = 0$ *for* $q \geq 2$ *or for* $q = 1$ *and* $n \geq 1$. *Furthermore,* $\dim R^1 \varphi_* O_X = 1$.

Proof. – By Proposition 1 and the remark made above, we have only to prove that $R^1 \varphi_* I^n = 0$ if $n \geq 1$; but we are sure that $R^1 \varphi_* I^n = 0$ for $n \gg 0$. From the exact sequence

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow i_* O_{X'}(ns) \longrightarrow 0,$$

we deduce the exact sequence

$$R^1 \varphi_*(I^{n+1}) \longrightarrow R^1 \varphi_*(I^n) \longrightarrow R^1 \varphi_*(i_* O_{X'}(ns)) = H^1(X', O_{X'}(ns)).$$

In view of (A) (in which we take $d = 1$, $m_1 = 3$, $N = 0$), $H^1(X', O_{X'}(ns))$ is the dual of the vector space $H^0(X', O_{X'}(-ns))$, which is zero if $n > 0$ and isomorphic with K if $n = 0$. This proves our proposition.

We wish to investigate now the following problem: let X′ be a projective variety, $i : X' \hookrightarrow X$ a closed immersion given by an invertible ideal, I, such that the conditions *a*) and *b*) are fulfilled; when does exist an algebraic variety Y with a proper morphism $\varphi : X \to Y$ such that X coincides with the blowing-up of Y at some normal point y and X′ with the exceptional locus of φ? When such a variety exists we say that X is contractible along X′ and Y will be referred to the contraction of X along X′. It follows immediately that this contraction is unique up to an isomorphism, if it exists.

PROPOSITION 3. – *Let* X′ *be a projective variety,* $i : X' \hookrightarrow X$ *a closed immersion given by an invertible sheaf of ideals* I, *such that* X *is normal in every point of* X′. *Assume that* $R^1 \psi_*(I^2) = 0$, *where* $\psi : X \to T$ *is the canonical continuous map into the quotient space* T *obtained by identifying the points of* X′. *Then, if the conditions a*) *and b*) *are fulfilled,* X *is contractable along* X′.

First of all we prove the following:

LEMMA. – *In the hypotheses stated above, there exists an open neighbourhood* U *of* X′ *in* X *such that the following conditions are fulfilled*:

i) *The homomorphism of restriction*

$$H^0(U, I) \longrightarrow H^0(X', O_{X'}(s))$$

is surjective;

ii) $p O_X = I$, *where* $p = H^0(U, I)$.

Proof. – Consider the commutative diagram (in the category of topological spaces):

$$\begin{array}{ccc} X' & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\psi} \\ \operatorname{Spec} K = \{y\} & \overset{j'}{\hookrightarrow} & T \end{array}$$

and the exact sequence

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow i_* O_{X'}(s) \longrightarrow 0$$

which gives rise to the exact sequence

$$\psi_*(I) \longrightarrow \psi_* i_* O_{X'}(s) \longrightarrow R^1\psi_*(I^2) = 0 .$$

In particular, we get the surjection

$$(*) \qquad (\psi_*(I))_y \longrightarrow (\psi_* i_* O_{X'}(s))_y .$$

On the other hand, $(\psi_*(I))_y = \varinjlim_{V \ni y} H^0(V, \psi_*(I)) = \varinjlim_{U \supset X'} H^0(U, I)$ and $(\psi_* i_* O_{X'}(s))_y = (j'_* \varphi'_* O_{X'}(s))_y = H^0(\{y\}, \varphi'_* O_{X'}(s)) = H^0(X', O_{X'}(s))$.

Since $H^0(X', O_{X'}(s))$ is a finite dimensional vector space over K, the surjectivity of (*) and the definition of the direct limit show the existence of a neighbourhood U satisfying i).

In order to prove ii), we remark that this assertion is equivalent to the one that the invertible sheaf I is generated by its sections on U, i.e. $U = \bigcup_f U_f$ $(f \in H^0(U, I))$, where $U_f = \{x \in U / f(x) \neq 0\}$. But since $i^* I = O_{X'}(s)$, we have $X' = \bigcup_{f'} X'_{f'}$ $[f' \in H^0(X', O_{X'}(s))]$. Let $x \in X'$ be a point and $f' \in H^0(X', O_{X'}(s))$ a section such that $x \in X'_{f'}$; by i), there exists a section $f \in H^0(X, I)$ such that $f' = i^*(f)$. Then $x \in X_f$, which implies that X_f contains a whole neighbourhood U_x of x. We finish the proof of ii) by taking for U the union of U_x with $x \in X'$.

Proof of Proposition 3. – We can now assume that the conditions of the lemma are fulfilled for X, replacing, if necessary, X by U, since the problem is local along X'. Consider the commutative diagram

$$\begin{array}{ccc} X' & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ \{y\} & \overset{j_1}{\hookrightarrow} & Y_1 \end{array}$$

where $Y_1 = \operatorname{Spec} H^0(X, O_X)$, φ_1 corresponds to the identity of $H^0(X, O_X)$, and the closed immersion j_1 to the surjective homomorphism $H^0(X, O_X) \to K$

[whose kernel is $p = H^0(X, I)$]. Let $\pi : X_1 = \mathrm{Proj} \bigoplus_{n \geq 0} p^n \to Y_1$ be the blowing-up of Y at y. Since $I = \varphi_1^{-1}(p) = p\,O_X$ is invertible, by the universal property of the blowing-up, we get a unique Y–morphism $\varepsilon : X \to X_1$, which corresponds to the canonical inclusion $\bigoplus_{n \geq 0} p^n \to \bigoplus_{n \geq 0} H^0(X, I^n)$, such that $\varepsilon^* O_X(1) = I$. In order to prove the required result, it is sufficient to show that $\varepsilon(X') = \pi^{-1}(y)$ and that ε is an open immersion in a certain open neighbourhood of X′.

The composed homomorphism of graded algebras

$$\bigoplus_{n \geq 0} p^n \longrightarrow \bigoplus_{n \geq 0} H^0(X, I^n) \longrightarrow \bigoplus_{n \geq 0} H^0(X', O_{X'}(sn))$$

(which corresponds to the morphism $\varepsilon \circ i$) is surjective, since its first component is so [condition i) of the lemma] and the graded algebra $\bigoplus_{n \geq 0} H^0(X', O_{X'}(sn))$ is generated by $H^0(X', O_{X'}(s))$ [by (A)]. It follows that the morphism $\varepsilon \circ i : X' \to X$ is a closed immersion, consequently the inclusion $\varepsilon \circ i(X') \subset \subset \pi^{-1}(y)$ implies $\dim X - 1 \leq \dim X_1 - 1$, or $\dim X \leq \dim X_1$; hence $\dim X = \dim X_1$, φ_1 being dominating. From the assumption that X is a birational morphism, therefore ε is birational too.

Since $\varepsilon \circ i$ is an immersion and $\varepsilon^{-1}(\pi^{-1}(y)) = \varphi_1^{-1}(y) = X'$, we have $\varepsilon^{-1}(\varepsilon(x)) = \{x\}$ for every $x \in X'$; hence every $x \in X'$ is isolated in its fibre with respect to ε. By Zariski's Main Theorem, there exists an open neighbourhood U_0 of X′ in X and an open immersion $\eta : U_0 \hookrightarrow Z$, with Z the normalisation of X, such that the following diagram

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \eta\ } & Z \\ \cap\downarrow & & \downarrow \xi \\ X & \xrightarrow{\ \varepsilon\ } & X \end{array}$$

is commutative (ξ is the canonical finite morphism). It is easily seen that $\eta(X')$ is a connected component of $(\pi \circ \xi)^{-1}(y)$ and, since Y_1 is normal, Zariski's Connectedness Theorem shows that $\eta(X') = (\pi \circ \xi)^{-1}(y)$, i.e. $\pi^{-1}(y) = = \varepsilon(X')$. Consequently $\xi/\eta(X')$ is a bijection and, since it is a finite morphism, for every $z \in \eta(X')$ the homomorphism of local rings

$$\xi_z^* : O_{X_1, \xi(z)} \longrightarrow O_{Z, z}$$

is finite; hence, for every $x \in X'$, the homomorphism

$$\varepsilon_x^* : O_{X_1, \varepsilon(x)} \longrightarrow O_{Z, z}$$

is finite. Finally, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p\,O_{X_1, \varepsilon(x)} & \longrightarrow & O_{X_1, \varepsilon(x)} & \longrightarrow & O_{\pi^{-1}(y), \varepsilon(x)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_x & \longrightarrow & O_{X, x} & \longrightarrow & O_{X', x} & \longrightarrow & 0 \end{array}$$

in which the lines are exact and the last vertical arrow is surjective, together with the assumption $p O_X = I$, show that

$$m_{X_1, \varepsilon(x)} O_{X,x} = m_{X,x} .$$

Therefore ε_x^* is an isomorphism, because the local rings $O_{X,x}$ and $O_{X_1,\varepsilon(x)}$ have the same residue fields and one can apply Nakayama's lemma. Hence ε is biregular in every point of X', which completes the proof.

THEOREM. – *Let* X' *be a projective variety, and* $i : X' \hookrightarrow X$ *a closed immersion given by an invertible sheaf of ideals* I, *such that* X *is projective and normal at every point of* X'. *Assume that there exists a suitable immersion* $h : X' \hookrightarrow P_n$ *such that* X' *is a complete intersection in* P_n *and the Picard group of* X' *is generated by the class of* $O_{X'}(1) = h^* O_{P_n}(1)$. *Then* X *is contractible along* X' *if, and only if,* $i^* I$ *is ample on* X' *and, in this case, the contraction* Y (*which is unique*) *is also projective.*

Remarks. – *a*) The condition that Pic X' is generated by the class of $O_{X'}(1)$ is always fulfilled if $\dim X' \geq 3$ [by (B)], as well as for $\dim X' = 2$ or $\dim X' = 1$ if $X' = P_2$ and $X' = P_1$ respectively.

b) In [1] a more general criterion for contractibility is given (and, in particular, a more general criterion for contractibility to a point): but the contraction is there only an algebraic space. On the contrary, the previous theorem gives sufficient conditions in order that the contraction is in fact an algebraic variety.

Proof. – We recall that, in the proof of Proposition 3, the cohomological condition $R^1 \psi_* (I^2) = 0$ has been used only for deducing the condition i) of the lemma [from which condition ii) follows restraining again X along X'].

On the other hand, it is easy to see that the condition "$i^* I$ is ample on X'" can be replaced by "$i^* I = O_{X'}(s)$ where $s > 0$".

We shall prove that the condition i) of the lemma is a consequence of the hypothesis of projectivity of X. Let then H be a hyperplane section on X (i.e. a very ample invertible sheaf on X) and therefore $i^* H = O_{X'}(m)$, where $m > 0$, since Pic X' is generated by $O_{X'}(1)$; further we can suppose m a multiple of s, replacing, if necessary, H by H^s: hence $m = st$ with $t \in Z$. For every integer N, we have the exact sequence:

$$0 \longrightarrow I^{N+1} \longrightarrow I^n \longrightarrow i_* O_{X'}(sN) \longrightarrow 0$$

[because $i^* I = O_{X'}(s)$]. If we choose a positive integer n sufficiently large such that $H^1(X, I(n)) = 0$, where $I(n) = I \otimes H^{\otimes n}$, we get by tensorising with $I(n)$ the exact sequence:

$$0 \longrightarrow I^{N+2}(n) \longrightarrow I^{N+1}(n) \longrightarrow i_* O_{X'}(s(N+1+tn)) \longrightarrow 0 ,$$

which gives rise to the exact sequence of cohomology:

$$H^1(X, I^{N+2}(n)) \longrightarrow H^1(X, I^{N+1}(n)) \longrightarrow H^1(X', O_{X'}(s(N+1+tn)) .$$

But $H^1(X', O_{X'}(a)) = 0$ if $a \geq 0$ since either $X' = P_1$ or, if $\dim X' \geq 2$, one can apply (A); hence $H^1(X, I(n)) = 0$ implies (by taking inductively $N = -1, -2, \cdots, -tn-1$) that $H^1(X, I^{-tn+2}(n)) = 0$ and $H^1(X, I^{-tn+1}(n)) = 0$. In other words, the homomorphisms of restriction:

$$H^0(X, I^{-tn+1}(n)) \longrightarrow H^0(X', O_{X'}(s))$$

$$H^0(X, I^{-tn}(n)) \longrightarrow H^0(X', O_{X'})$$

are surjective. From the second surjection it follows the existence of a section $\alpha \in H^0(X, I^{-tn}(n))$ such that $i^*(\alpha) = 1$, and, if D is the Cartier divisor of α, we have $X' \cap \operatorname{Supp} D = \varnothing$, i.e., $X' \subset U = X - \operatorname{Supp} D$; then D/U is linearly equivalent with the zero divisor and therefore $I^{-tn+1}(n)/U = I/U$. The first surjection and the commutative diagram

$$\begin{array}{ccc} H^0(X, I^{-tn+1}(n)) & \searrow & \\ \text{res} \downarrow & & H^0(X', O_{X'}(s)) \\ H^0(U, I) & \nearrow & \end{array}$$

prove the existence of the contraction Y (which is unique).

We have now only to prove that Y is projective. But $I^{-nt}(n)/X - X'$ is very ample on $X - X'$, since $I^{-tn}(n)/X - X' = H^{\otimes n}/X - X'$.

On the other hand, if α is the section considered above, the canonical rational map $u : X \to P(H^0(X, I^{-tn}(n)))$ is every where defined and it is an isomorphism between $X - X'$ and $u(X - X')$, where $P(H^0(X, I^{-tn}(n)))$ is the projective space associated to the finite dimensional vector space $H^0(X, I^{-tn}(n))$. But $u(X')$ is a single point, since, if one chooses a basis $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_m$ of this vector space such that $i^*(\alpha_i) = 0$ for $i \geq 2$, then $u(X') = (1, 0, \cdots, 0) = y'$. The projectivity of X implies that $u(X)$ is also projective. One can suppose $u(X)$ normal in y', replacing, if necessary, $u(X)$ by its normalisation (which remains projective). Then the contraction must be isomorphic with $u(X)$, and so Y is projective, which completes the proof of the theorem.

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