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Analisi matematica. — *Exponential stability of difference equations which cannot be linearized.* Nota di FRANCESCO S. DE BLASI (*) e JOHN SCHINAS (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si considera l'equazione $\Delta x(t) = f(x(t-1))$ e si dimostra che, se f ha differenziale multivoco D_f in $x = 0$ e tutte le soluzioni di $\Delta x(t) \in D_f(x(t-1))$ tendono all'origine, allora quest'ultima è localmente esponenzialmente stabile per l'equazione data.

1. It is well known (see [2] Ch. V, § 43) that if $f: E^n \to E^n$ is continuously differentiable, with Fréchet differential A at $x = 0$, and if all solutions of the linear difference equation

$$\Delta x(t) = Ax(t-1) \quad , \quad \Delta x(t) = x(t) - x(t-1), \tag{1}$$

approach zero as $t \to \infty$, the origin is locally exponentially stable for

$$\Delta x(t) = f(x(t-1)). \tag{2}$$

The aim of this paper is to extend the previous result to the case in which f is not necessarily Fréchet differentiable at the origin. For differential equations such problem has been treated in a recent work by Lasota and Strauss ([3]), who have introduced for this purpose the concept of multivalued differential. The definition of the multivalued differential D_f of f, that we shall use, is essentially the same with the difference that $D_f(x), x \in E^n$, will be required to be a nonempty compact subset of E^n without the additional hypothesis of convexity, which occurs in [3]. If f has Fréchet differential A at the origin, we have $D_f(x) = \{Ax\}, x \in E^n$. We shall prove the following generalization of the aforementioned result:

If f has multivalued differential D_f at $x = 0$ (see next paragraph) and if all solutions of the multivalued difference equation

$$\Delta x(t) \in D_f(x(t-1)) \tag{3}$$

approach zero as $t \to \infty$, then the origin is locally exponentially stable for equation (2).

The proof of this result can be described as the discrete analogue of a corresponding one, devised by Lasota and Strauss in the case of ordinary differential equations. It actually depends on certain perturbation theorems

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for multivalued difference equations, which have been established in [1]. Other applications of multivalued difference equations can be found in [4].

2. Denote by: $N_{t_0} = \{t_0, t_0 + 1, \cdots\}$, where t_0 is any natural number or zero; E^n the n-dimensional real Euclidean space with norm $|\cdot|$; $B(r)$ the closed ball with center the origin of E^n and radius $r \geq 0$; $\|X\| = \sup\{|x| : x \in X\}$, where X is a nonempty and bounded subset of E^n; K^n the family of all nonempty compact subsets of E^n. In K^n addition and multiplication by nonnegative scalars are defined by $X + Y = \{x + y : x \in X, y \in Y\}$, $\lambda X = \{\lambda x : x \in X\}$. We shall denote by Φ the family of all uppersemicontinuous functions $F : E^n \to K^n$ and by χ the subfamily of Φ consisting of all homogeneous functions, i.e. of all F such that $F(\lambda x) = \lambda F(x)$, for all $x \in E^n$ and $\lambda \geq 0$.

DEFINITION 1. Let $F : E^n \to K^n$. We say that F is locally Lipschitz at $x = 0$ if there exist positive constants L and δ such that

$$\|F(x)\| \leq L|x| \qquad \text{for all} \quad |x| \leq \delta .$$

If $\delta = \infty$, F is called globally Lipschitz at $x = 0$.

DEFINITION 2. Let $F \in \Phi$ be locally (globally) Lipschitz at $x = 0$. A function $\varphi \in \chi$ is called a local (global) upper differential of F if there exists a $\delta > 0$, $(\delta = \infty)$ such that

$$F(x) \subset \varphi(x) \qquad \text{for all} \quad |x| \leq \delta \quad (\text{for all } x \in E^n).$$

Note that if $F \in \Phi$ is locally (globally) Lipschitz at $x = 0$, $\varphi(x) = LB(|x|)$ is a local (global) upper differential of F.

DEFINITION 3. Let $F \in \Phi$ be locally Lipschitz at $x = 0$. We define the multivalued differential D_F of F by

$$D_F(x) = \cap\{\varphi(x) : \varphi \text{ is a local upper differential of } F\}, \qquad x \in E^n.$$

The multivalued differential D_F^* of a function F which is globally Lipschitz at $x = 0$, is defined by

$$D_F^*(x) = \cap\{\varphi(x) : \varphi \text{ is a global upper differential of } F\}, \qquad x \in E^n.$$

It is clear that the preceding definitions apply in particular to single valued functions $f : E^n \to E^n$. For the function $f : E^2 \to E^2$ given by $\left(\frac{1}{2}(x_1^2 + x_2^2)^{1/2} \sin(x_1^2 + x_2^2)^{-1/2}, \frac{1}{2}(x_1^2 + x_2^2)^{1/2}\right)$, $(x_1, x_2) \in E^2$, one can easily verify that

$$D_f(x_1, x_2) = D_f^*(x_1, x_2) = \begin{pmatrix} \left[-\frac{1}{2}(x_1^2 + x_2^2)^{1/2}, \frac{1}{2}(x_1^2 + x_2^2)^{1/2}\right] \\ \frac{1}{2}(x_1^2 + x_2^2)^{1/2} \end{pmatrix}, \qquad (x_1, x_2) \in E^2.$$

Using the same argument as in Lemma 2.8 of [3], one can prove the following

LEMMA 1. *Let* $F \in \Phi$ *be locally (globally) Lipschitz at* $x = 0$. *Then* $D_F \in \chi$ $(D_F^* \in \chi)$. *Furthermore, there exists a sequence* $\{\varphi_k\}$ *of local (global) upper differentials such that*

$$\varphi_{k+1}(x) \subset \varphi_k(x) \qquad \textit{for every} \quad x \in E^n \qquad \textit{and} \quad k = 1, 2, \cdots$$

$$D_F(x) = \bigcap_{k=1}^{\infty} \varphi_k(x) \quad , \quad \left(D_F^*(x) = \bigcap_{k=1}^{\infty} \varphi_k(x)\right), \qquad x \in E^n.$$

Note that if $F \in \Phi$ is globally Lipschitz at $x = 0$, we have $D_F(x) \subset \subset D_F^*(x)$, $x \in E^n$.

DEFINITION 4. Assume that $f: E^n \to E^n$ is continuous and locally Lipschitz at $x = 0$. The function $h: E^n \to E^n$ is called a homogeneous differential of f at $x = 0$, if h is homogeneous and continuous and

$$|f(x) - h(x)| = o(|x|) \qquad \text{as} \quad |x| \to 0.$$

The homogeneous differential is unique ([3]).

LEMMA 2. *Assume that* $f: E^n \to E^n$ *is continuous and locally Lipschitz at* $x = 0$. *If* f *has homogeneous differential* h, *then* D_f *is single valued and we have* $D_f(x) = \{h(x)\}$, $x \in E^n$; *conversely if* D_f *is single valued*, f *has homogeneous differential* h *and* $D_f(x) = \{h(x)\}$, $x \in E^n$. *In particular* f *is Fréchet differentiable if and only if, for some matrix* A, $D_f(x) = \{Ax\}$, $x \in E^n$.

The proof of Lemma 2 is given in [3].

3. Consider the multivalued difference equation

$$\Delta x(t) \in F(x(t-1)). \tag{4}$$

DEFINITION 5. Let $t_0 \in N_0$, $x_0 \in E^n$. A function $x: N_{t_0} \to E^n$ is called solution of (4) if $x(t_0) = x_0$ and $x(t)$ satisfies (4) for all $t \in N_{t_0+1}$.

Note that, for any $t_0 \in N_0$ and $x_0 \in E^n$, (4) has at least one solution $x(t)$, with $x(t_0) = x_0$.

To prove our main results we shall use the following Lemmas which can be found in [1].

LEMMA 3. *Suppose that*:

(i) $\{F_k\}$ *is an infinite sequence of functions in* χ *such that* $F_{k+1}(x) \subset \subset F_k(x)$, *for all* $x \in E^n$, $k \in N_1$, *and define* $F(x) = \bigcap_{k=1}^{\infty} F_k(x)$;

(ii) *all solutions* $x(t)$ *of* (4), *with* $x(0) \in E^n$, *approach zero as* $t \to \infty$.

Then there exist $k \in N_1$ *and* $L \geq 1$ *such that every solution* $x(t)$ *of*

$$\Delta x(t) \in F_k(x(t-1)), \tag{5_k}$$

with $x(0) \in E^n$, *satisfies* $|x(t)| \leq L|x(0)|$, *for all* $t \in N_0$.

LEMMA 4. *Let* $F \in \chi$. *Suppose that there exist* $\varepsilon > 0$ *and* $H \geq 1$ *such that any solution* $x(t)$ *of*

$$\Delta x(t) \in F(x(t-1)) + \varepsilon B(|x(t-1)|),$$

with $x(0) \in E^n$, *satisfies* $|x(t)| \leq H|x(0)|$, $t \in N_0$. *Let* $0 \leq \sigma < \varepsilon$. *Then, for every solution* $x(t)$ *of*

$$\Delta x(t) \in F(x(t-1)) + \sigma B(|x(t-1)|),$$

with $x(0) \in E^n$, *we have* $|x(t)| \leq H|x(0)|\rho^{-t}$, $t \in N_0$, $1 < \rho \leq 1 + (\varepsilon - \sigma)H^{-1}$.

LEMMA 5. *Suppose that:*

(i) *every solution* $x(t)$ *of* (4), *where* $x(0) \in E^n$ *and* $F \in \chi$, *approaches zero as* $t \to \infty$;

(ii) $G \in \Phi$ *is such that* $\|G(x)\| = o(|x|)$ *as* $|x| \to 0$.

Then there exist constants $\delta > 0$, $M \geq 1$ *and* $\rho > 1$ *such that, for any solution* $x(t)$ *of*

$$\Delta x(t) \in F(x(t-1)) + G(x(t-1)),$$

if $|x(0)| < \delta$, *we have* $|x(t)| \leq M|x(0)|\rho^{-t}$, $t \in N_0$.

4. This paragraph contains our main results.

THEOREM 1. *Let* $F \in \Phi$ *be locally Lipschitz at* $x = 0$. *If all solutions* $x(t)$ *of*

$$\Delta x(t) \in D_F(x(t-1)),$$

with $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *then there exist constants* $\delta > 0$, $M \geq 1$ *and* $\rho > 1$ *such that all solutions* $x(t)$ *of* (4), *with* $|x(0)| < \delta$, *satisfy* $|x(t)| \leq M|x(0)|\rho^{-t}$, $t \in N_0$.

Proof. Let $\{\varphi_k\}$ be the sequence which corresponds to D_F according to Lemma 1 and define

$$F_k(x) = \varphi_k(x) + \frac{1}{k} B(|x|), \quad x \in E^n, \quad k \in N_1.$$

Observe that for all $k \in N_1$, $F_k \in \chi$, $F_{k+1}(x) \supset F_k(x)$ and $\bigcap_{k=1}^{\infty} F_k(x) = D_F(x)$. From Lemma 3 there exist $k \in N_1$ and $L \geq 1$ such that every solution $x(t)$ of

$$\Delta x(t) \in \varphi_k(x(t-1)) + \frac{1}{k} B(|x(t-1)|),$$

with $x(0) \in E^n$, satisfies $|x(t)| \leq L|x(0)|$, $t \in N_0$. Applying Lemma 4 to this equation and choosing $\sigma = 0$, we can find that all solutions $x(t)$ of

$$\Delta x(t) \in \varphi_k(x(t-1)), \tag{6}$$

with $x(0) \in E^n$, approach zero exponentially as $t \to \infty$. From Definition 2 there exists a $\delta_k > 0$ such that $F(x) \subset \varphi_k(x)$ if $|x| \leq \delta_k$. Define $G : E^n \to K^n$ by:

$$G(x) = \begin{cases} \{0\}, & \text{if } |x| < \delta_k \\ B(\|F(x)\| + \|\varphi_k(x)\|), & \text{if } |x| \geq \delta_k. \end{cases}$$

Clearly

$$F(x) \subset \varphi_k(x) + G(x) \quad \text{for all} \quad x \in E^n. \tag{7}$$

Moreover $G \in \Phi$ and $\|G(x)\| = o(|x|)$ as $x \to 0$. So, from Lemma 5, there exist constants $\delta > 0$, $M \geq 1$ and $\rho > 1$ such that, for any solution $x(t)$ of

$$\Delta x(t) \in \varphi_k(x(t-1)) + G(x(t-1)),$$

if $|x(0)| < \delta$, we have $|x(t)| \leq M|x(0)|\rho^{-t}$, $t \in N_0$. Because of (7) this conclusion holds in particular for all solutions $x(t)$ of (4) with $|x(0)| < \delta$. This completes the proof.

THEOREM 2. *Let* $F \in \Phi$ *be globally Lipschitz at* $x = 0$. *If all solutions* $x(t)$ *of*

$$\Delta x(t) \in D_F^*(x(t-1)),$$

with $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *then there exist constants* $H \geq 1$ *and* $\rho > 1$ *such that solutions* $x(t)$ *of* (4), *with* $x(0) \in E^n$, *satisfy* $|x(t)| \leq H|x(0)|\rho^{-t}$, $t \in N_0$.

Proof. Denote by $\{\varphi_k\}$ the sequence wich corresponds to D_F^* according to Lemma 1. By the same argument of Theorem 1 we find that all solutions $x(t)$ of (6), with $x(0) \in E^n$, satisfy $|x(t)| \leq H|x(0)|\rho^{-t}$, $t \in N_0$, where $H \geq 1$ and $\rho > 1$ are the constants in Lemma 4. The last inequality is in particular true for all solutions $x(t)$ of (4), with $x(0) \in E^n$, since $\varphi_k(x) \supset F(x)$ for all $x \in E^n$.

When F is single valued from the preceding Theorems we have:

COROLLARY 1. *Let* $f : E^n \to E^n$ *be continuous and locally Lipschitz at* $x = 0$. *If all solutions* $x(t)$ *of* (3), *with* $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *then there exist constants* $\delta > 0$, $M \geq 1$ *and* $\rho > 1$ *such that all solutions* $x(t)$ *of* (2), *with* $|x(0)| < \delta$, *satisfy* $|x(t)| \leq M|x(0)|\rho^{-t}$, $t \in N_0$.

COROLLARY 2. *Let* $f : E^n \to E^n$ *be continuous and globally Lipschitz at* $x = 0$. *If all solutions* $x(t)$ *of* $\Delta x(t) \in D_f^*(x(t-1))$, *with* $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *then there exist constants* $H \geq 1$ *and* $\rho > 1$ *such that all solutions* $x(t)$ *of* (2), *with* $x(0) \in E^n$, *satisfy* $|x(t)| \leq H|x(0)|\rho^{-t}$, $t \in N_0$.

From Lemma 2 and Corollary 1 we have:

THEOREM 3. *Let* $f : E^n \to E^n$ *be continuous and locally Lipschitz at* $x = 0$ *and assume that* f *has homogeneous differential* h *at* $x = 0$.

If all solutions $x(t)$ *of*

$$\Delta x(t) = h(x(t-1)), \tag{8}$$

with $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *then there exist constants* $\delta > 0$, $M \geq 1$ *and* $\rho > 1$ *such that all solutions* $x(t)$ *of* (2), *with* $|x(0)| < \delta$, *satisfy* $|x(t)| \leq M\,|x(0)|\,\rho^{-t}$, $t \in N_0$.

COROLLARY 3. *Let* $f: E^n \to E^n$ *be continuous and locally Lipschitz at* $x = 0$ *and assume that* f *has Fréchet differential* A *at* $x = 0$. *Then, if all solutions* $x(t)$ *of* (1), *with* $x(0) \in E^n$, *satisfy* $x(t) \to 0$ *as* $t \to \infty$, *the conclusion of Theorem* 3 *holds.*

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