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On the boundedness of the motions of a periodic process

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Equazioni differenziali. — *On the boundedness of the motions of a periodic process.* Nota di NICOLAE PAVEL, presentata [(*)] dal Socio G. SANSONE.

RIASSUNTO. — Riallacciandosi alla teoria delle equazioni differenziali non autonome l'Autore discute alcuni concetti di limitatezza del moto di un processo periodico in uno spazio di Banach.

1. INTRODUCTION

The basic theory of the periodic processes on a Banach space is developed by J. K. Hale, G. P. La Salle, J. E. Billotti and M. Slemrod in [2], [6].

The concept of process on a Banach space defined by above authors includes ordinary differential equations, functional differential equations of retarded type, some systems arising in the theory of elasticity, etc.

In this paper, the concepts of the boundedness of the motions of a process defined in [2], [6], in the same spirit of Levinson [8] and Yoshizawa [18] are defined.

In the case of a finite dimensional space we prove that every dissipative periodic process is uniformly bounded and uniformly ultimately bounded.

From a result of [2], [6], it follows that every ω-periodic process on E^n has at least one ω-periodic motion (it is used [1]).

It seems that in the case of a general Banach space the above results are not true.

However, in § 3 (using a result of [2], [6]) we prove that if the associated mapping T of a periodic process on a Banach space is dissipative, then this process is uniformly bounded and uniformly ultimately bounded.

These results extend some results of [11], [15], [18], to periodic processes on Banach spaces.

Taking into account the result of C. C. Conley and R. K. Miller in [3] and Remark 2.2 in [11], it follows that in the case of an almost periodic process on E^n, ultimate boundedness does not necessarily imply uniform boundedness.

In the case of a process defined by an ordinary differential equation, more complete results and references can be found in the books of V. A. Pliss [15], R. Ressig, G. Sansone, R. Conti [16] and T. Yoshizawa [18].

(*) Nella seduta del 13 gennaio 1973.

2. Definition and Preliminary

Let X be a Banach space, R the real numbers and $R_+ = [0, +\infty)$.

Consider a mapping $u : R \times R_+ \times X$ and define for $\sigma \in R$, $\tau \in R_+$ the operator $U(\sigma, \tau) : X \to X$ by

$$U(\sigma, \tau) x = u(\sigma, \tau, x) . \tag{2.1}$$

The following definitions are given in [6], [2].

Definitions 2.1. A process on X is a mapping $u : R \times R_+ \times X \to X$ with the properties:

(2.2) u is continuous

(2.3) $U(\sigma, 0) = I$, where I is the identity

(2.4) $U(\sigma + s, \tau) \, U(\sigma, s) = U(\sigma, s + \tau)$, for all $\sigma \in R$, $s, \tau \in R_+$.

The positive motion through (σ, x) is the set $\{u(\sigma, \tau, x), \tau \in R_+\}$. A motion is said to be periodic of period $\alpha > 0$, if $U(\sigma, \tau + \alpha) = U(\sigma, \tau)$ for all $\tau \in R_+$.

A process u is said to be periodic of period ω (or ω-periodic) if $U(\sigma + \omega, \tau) = U(\sigma, \tau)$ for all $\sigma \in R$, $\tau \in R_+$. If $U(\sigma, \tau) = U(0, \tau)$ for all $\sigma \in R$, $\tau \in R_+$, then u is said to be autonomous (or dynamical system).

A large number of examples of such processes on a Banach space X can be found in [6].

We recall only the following example of processes.

Let E^n be Euclidean n-space and $f : R \times E^n \to E^n$ a continuous function such that the initial value problem (2.5) + (i) has a unique solution $\varphi(t, t_0, x_0)$, $t \geq t_0$, $\varphi(t_0, t_0, x_0) = x_0$, where

$$\frac{dx}{dt} = f(t, x) \tag{2.5}$$

$$x(t_0) = x_0 , \quad t_0 \in R \quad , \quad x_0 \in E^n. \tag{i}$$

Define u: $R \times R_+ \times E^n \to E^n$ by $u(\sigma, \tau, x) = \varphi(\sigma + \tau, \sigma, x)$.

It is easy to verify that u is a process on E^n.

For a ω-periodic process u on X let us consider for any fixed $\sigma \in R$ the continuous mapping $T : X \to X$ defined by

$$Tx = u(\sigma, \omega, x) = U(\sigma, \omega) x. \tag{2.6}$$

T is called in [6] "the associated mapping of u".

It follows that $T^n = U(\sigma, n\omega)$, where T^n is the n-th iterate of T.

Definition 2.2. a) T is said to be smooth if there is a non negative integer n_0 such that for each bounded set B in X there is a compact set B′

in X such that $T^n x \in B$ for $n = 0, 1, \cdots, N$ ($N \geq n_0$) implies $T^n x \in B$ for $n = n_0, n_0 + 1, \cdots, N$.

b) T is said to be dissipative if it is smooth and if there is a bounded set $B \subset X$ with the property that given $x \in X$ there is a positive integer $N(x)$ such that $T^n x \in B$ for $N(x) \leq n \leq N(x) + n_0$ (see [2], [6]).

In [2] and [6] are established the following results.

THEOREM 2.1. *If* T *is dissipative, then there is a compact* K *in* X *with the property that given a compact* H *in* X *there is a positive integer* N(H) *and an open neighbourhood* O_H *of* H *such that* $T^n(O_H) \subset K$ *for all* $n \geq N(H)$.

COROLLARY 2.1. *If* T *is dissipative and maps bounded sets into bounded sets, then* T^j *has a fixed point for each integer* $j \geq n_0$.

Remark 2.1. Theorem 2.1 generalizes Theorems 2.1 and 2.2 of [15]. Corollary 2.1 generalizes some results of [19], [4] and Theorem 2.4 of [11]. (For the ordinary differential systems (2.5), $n_0 = 1$).

Of course, the integers $N(x)$ and $N(H)$ defined above, depend on σ fixed in R (σ appearing in (2.6)).

Now, in a similar way as in [8], [18] we define the following concepts of boundedness of a process u on a Banach space X.

DEFINITION 2.3. u is called equibounded ($e \cdot b$) if for any compact set H in X and $\sigma \in R$, there is a compact $H'(H, \sigma)$ in X such that $u(\sigma, \tau, x) \in H'$ for all $\tau \in R_+$ and $x \in H$.

DEFINITION 2.4. u is said to be uniformly bounded ($u \cdot b$) if for any compact H in X, there is a compact $H'(H)$ in X such that $u(\sigma, \tau, x) \in H'$ for all $\sigma \in R$, $\tau \in R_+$ and $x \in H$. (i.e. H' appearing in Definition 2.3 may be chosen independent of σ).

DEFINITION 2.5. u is called preultimately bounded ($p \cdot u \cdot b$) or predissipative if there is a bounded set B in X with the property that for each $\sigma \in R$, $x \in X$ there is $\tau_0(\sigma, x) > 0$, such that $u(\sigma, \tau_0, x) \in B$.

DEFINITION 2.6. u is called ultimately bounded ($u \cdot b$; d) or dissipative if there is a bounded set $B \subset X$ with the property that for each $\sigma \in R$ and $x \in X$, there is $\tau_0(\sigma, x) > 0$ such that $u(\sigma, \tau, x) \in B$ for all $\tau \geq \tau_0$.

DEFINITION 2.7. u is used to be equiultimately bounded ($e \cdot u \cdot b$) if there is a bounded set B_0 in X with the property that given a compact H in X and $\sigma \in R$, there is $\tau_0(H, \sigma) > 0$, such that $u(\sigma, \tau, x) \in B_0$, for all $\tau \geq \tau_0$, $x \in H$.

DEFINITION 2.8. u is said to be uniformly ultimately bounded ($u \cdot u \cdot b$) if τ_0 appearing in Definition 2.7 may be chosen independent of σ.

Remark 2.2. It is known that in the case of the system (2.5) these concepts are different [18].

In the case of the initial value problem (2.5) + (i), the author proved the following results [11], [14].

THEOREM 2.2. *Any ultimately bounded periodic system is uniformly bounded* [10].

THEOREM 2.3. *Any ultimately bounded periodic system is uniformly ultimately bounded.*

THEOREM 2.4. *Any* $(u \cdot b\, ; d)$ ω*-periodic system has at least a* ω*-periodic solution.*

Remark 2.3. Taking into account Theorem 2.3 and Theorem 2.1 of Pliss [15], it follows that in the case of a periodic system (2.5), the concepts of $(p \cdot u \cdot b)$, $(u\, , b\, ; d)$, $(e \cdot u \cdot b)$ and $(u \cdot u \cdot b)$ are equivalent.

Obviously Corollary 2.1 generalizes Theorem 2.4.

3. BOUNDEDNESS OF THE PERIODIC PROCESSES ON X

The aim of this section is to show some consequences of Theorem 2.1. Namely we shall prove the following theorems.

THEOREM 3.1. *If the associated mapping* T *of a periodic process* u *is dissipative, then* u *is uniformly bounded.*

THEOREM 3.2. *If the associated mapping* T *of a periodic process* u *is dissipative, then* u *is uniformly ultimately bounded.*

Remark 3.1. Theorems 3.1 and 3.2 generalize Theorems 2.2 and 2.3 respectively.

Before proving the Theorems 3.1 and 3.2 we shall need two lemmas.

Let u be a ω-periodic process on X.

LEMMA 3.1. *In the case of the periodic process* u, *the concepts of* $(e \cdot b)$ *and* $(u \cdot b)$ *are equivalent.*

Proof. It is sufficient to prove that if u is $(e \cdot b)$ then u is $(u \cdot b)$. Let H be a compact set in X.

Since u is ω-periodic we may consider only the case $\sigma \in [0\, , \omega]$.

Let σ fixed in $[0\, , \omega)$. From (2.4) we derive

$$u(\sigma\, , t - \sigma\, , x) = u(\omega\, , t - \omega\, , u(\sigma\, , \omega - \sigma\, , x))\, , \quad \text{for all } t \geq \omega \text{ and } x \in X. \tag{3.1}$$

Set $\tilde{H} = \{u(t\, , \tau\, , x)\, ,\ t \in [0\, , \omega]\, ,\ \tau \in [0\, , \omega]\, ,\ x \in H\}$.

It follows $u(\sigma\, , \omega - \sigma\, , x) \in \tilde{H}$, for all $x \in H$.

But there is $H'(\tilde{H})$ such that $u(\omega\, , \tau\, , y) \in H'$ for all $\tau \geq 0$, $y \in \tilde{H}$ and therefore (from (3.1)) we have

$$u(\sigma\, , t - \sigma\, , x) \in H' \qquad \text{for all } t \geq \omega\, ,\ x \in H\, . \tag{3.2}$$

For $\sigma \leq t \leq \omega$ we have $u(\sigma\, , t - \sigma\, , x) \in \tilde{H}$ for all $x \in H$.

Since $\tilde{H} \subset H'$, we obtain

$$(3.3) \qquad u(\sigma, t-\sigma, x) \in H' \quad \text{for all} \quad \sigma \in [0, \omega], \quad t \geq \sigma, \quad x \in H.$$

Since H' depends only on H, the lemma is proved.

LEMMA 3.2. *In conditions of Lemma 1, the concepts of* $(e \cdot u \cdot b)$ *and* $(u \cdot u \cdot b)$ *are equivalent.*

Proof. Let u be $(e \cdot u \cdot b)$ and let H be a compact set in X.

Let $\tilde{H}(H) = \{u(t, \tau, x), \; t \in [0, \omega], \; \tau \in [0, \omega], \; x \in H\}$. With $\sigma \in [0, \omega]$, we have $u(\sigma, \omega - \sigma, x) \in \tilde{H}$, for all $x \in H$. By hypothesis there is $\tau_0(\tilde{H}) > 0$ such that

$$(3.4) \qquad u(\omega, \tau, y) \in B_0 \quad \text{for all} \quad \tau \geq \tau_0(\tilde{H}), \quad y \in \tilde{H}.$$

Using (3.1) and (3.4) we derive

$$(3.5) \qquad u(\sigma, t-\sigma, x) \in B_0 \quad \text{for all} \quad t \geq \omega + \tau_0, \quad x \in H$$

and therefore

$$(3.6) \qquad u(\sigma, \tau, x) \in B_0 \quad \text{for all} \quad \tau \geq \omega + \tau_0.$$

Since τ_0 is independent of σ, u is $(u \cdot u \cdot b)$ and lemma is proved.

Proof of Theorem 3.2. Suppose H is an arbitrary compact set in X and let $\sigma \in [0, \omega)$. By Theorem 2.1 there is a positive integer $N(H, \sigma)$ such that $T^n(H) \subset K$ for all $n \geq N(H, \sigma)$.

For $\tau \geq \omega N$ there is an integer $p \geq N(H, \sigma)$ and $\tau_1 \in [0, \omega]$ such that $\tau = p\omega + \tau_1$. Therefore

$$(3.7) \qquad u(\sigma, \tau, x) = u(\sigma, \tau_1, u(\sigma, p\omega, x)).$$

But $u(\sigma, p\omega, x) = T^p x \in K$ for all $x \in H$.

Let $\tilde{K} = \{u(t, \tau, y), \; t \in [0, \omega], \; \tau \in [0, \omega], \; y \in K\}$.

It follows (from (3.7))

$$(3.8) \qquad u(\sigma, \tau, x) \in \tilde{K} \quad \text{for all} \quad \tau \geq N(H, \sigma), \; x \in H$$

i.e. u is $(e \cdot u \cdot b)$ and therefore (by Lemma 3.2) u is $(u \cdot u \cdot b)$.

The proof of Theorem 3.2 is immediate.

Indeed, with the same notations as above, if

$$\tilde{H}(H, \sigma) = \{u(t, \tau, x), \; t \in [0, \omega], \; \tau \in [0, \omega N(H, \sigma)], \; x \in H\},$$

taking into account (3.8) we obtain

$$(3.9) \qquad u(\sigma, \tau, x) \in \tilde{K} \cup H(\tilde{H}, \sigma) \quad \text{for all} \quad \tau \in R_+, \; x \in H$$

i.e. u is $(e \cdot b)$ and therefore (by Lemma 3.1) u is $(u \cdot b)$.

Theorem is thus proved.

Remark 3.2. The Lemmas 3.1 and 3.2 generalize the Theorems 9.2, din 9.3 of [19].

The Theorem 4.2 in [11] is equivalent to a result of [4] so that is not new.

One suspects that u being $(u \cdot b\,; d)$ and periodic does not necessarily imply that u is $(u \cdot b)$ and $(u \cdot u \cdot b)$ (i.e. it seems that Theorems 2.2, 2.3 and 2.4 are not true in the case of a general Banach space X).

4. Periodic processes on E^n

In this section we shall prove the following results.

THEOREM 4.1. *Every preultimately bounded periodic process on* E^n *is uniformly bounded.*

THEOREM 4.2. *Every preultimately bounded periodic process on* E^n *is uniformly ultimately bounded.*

Remark 4.1. The fact that any preultimately bounded ω-periodic process on E^n has at least one ω-periodic motion, follows from Corollary 2.1 of [6] (or Corollary 2.2 of [2]) and Theorem 4.2.

Remark 4.2. Taking into account Theorem 4.2 it follows that in the case of a periodic process on E^n the concepts of $(p \cdot u \cdot b)$, $(u \cdot b\,; d)$, $(e \cdot u \cdot b)$ and $(u \cdot u \cdot b)$ are equivalent.

It follows that Theorem 4.2 generalizes Theorem 2.1 of Pliss [15] and Theorem 2.3.

Theorem 4.1 generalizes Theorem 2.2.

Proof of Theorem 4.1. Let u be a $(p \cdot u \cdot b)$ process on E^n (ω-periodic). Suppose that u is not $(u \cdot b)$ on E^n. Therefore, that is a compact set H_0 in E^n with the property that for each compact H in E^n there exist $\sigma_H \in R$, $\tau_H \in R_+$ and $x_H \in H_0$ such that $u(\sigma_H, \tau_H, x_H) \bar{\in} H$. Let us consider the sequence of the compact sets $H_p \subset E^n$ with the properties

$$(4.1) \qquad H_0 \cup B \subset A \subset H_1 \subset H_2 \cdots \subset H_p \subset H_{p+1} \subset \cdots$$

$$(4.2) \qquad u(t, \tau, x) \in H_p \qquad \text{for} \quad t \in [0, \omega], \quad \tau \in [0, p], \quad x \in A$$

where A is an arbitrary bounded set containing the union $H_0 \cup B$ and B is the bounded set appearing in Definition 2.5.

Denoting

$$\sigma_{H_p} = \sigma_p \quad , \quad \tau_{H_p} = \tau_p \quad , \quad x_{H_p} = x_p \, , \qquad p = 1, 2, \cdots$$

it follows

$$(4.3) \qquad u(\sigma_p, \tau_p, x_p) \bar{\in} H_p \, , \quad x_p \in H_0 \, , \quad \tau_p > 0 \, , \qquad p = 1, 2, \cdots$$

Let us consider the continuous function $g : R_+ \to E^n$ defined by $g(t) = u(\sigma_p, t, x_p)$.

We have $g(0) \in A$, $g(\tau_p) \bar{\in} A$. Set $\theta_p = \sup \{\theta \in [0, \tau_p], g(\theta) \in A\}$. It follows

$$(4.4) \qquad u(\sigma_p, \theta_p, x_p) \in A, \qquad 0 \leq \theta_p < \tau_p, \qquad p = 1, 2, \cdots$$

$$(4.5) \qquad u(\sigma_p, \tau, x_p) \bar{\in} A \quad \text{for} \quad \theta_p < \tau \leq \tau_p, \qquad p = 1, 2, \cdots$$

There is a unique integer m_p such that

$$(4.6) \qquad \sigma_p + \theta_p = m_p \omega + \bar{\theta}_p, \qquad 0 \leq \bar{\theta}_p \leq \omega.$$

Let $\bar{\tau}_p = \sigma_p + \tau_p - m_p \omega$, $p = 1, 2, \cdots$, therefore $\bar{\tau}_p = \bar{\theta}_p - \theta_p + \tau_p$. In as much as u is ω-periodic and using (2.4), we derive

$$(4.7) \qquad u(\bar{\theta}_p, t - \bar{\theta}_p, \bar{x}_p) = u(\sigma_p, t + \theta_p - \bar{\theta}_p, x_p), \quad t \geq \bar{\theta}_p, \quad p = 1, 2, \cdots$$

where $\bar{x}_p = u(\sigma_p, \theta_p, x_p) \in A$.

Because $\bar{\theta}_p < t \leq \bar{\tau}_p$ implies $\theta_p < t + \theta_p - \bar{\theta}_p \leq \tau_p$, taking into account (4.3), (4.5) and (4.7) we obtain

$$(4.8) \qquad u(\bar{\theta}_p, \bar{\tau}_p - \bar{\theta}_p, \bar{x}_p) \bar{\in} H_p \qquad p = 1, 2, \cdots$$

$$(4.9) \qquad u(\bar{\theta}_p, t - \bar{\theta}_p, \bar{x}_p) \bar{\in} A, \quad \text{for} \quad \bar{\theta}_p < t \leq \bar{\tau}_p, \qquad p = 1, 2, \cdots$$

Without loss of generality we may assume that the bounded sequences $\{\bar{\theta}_p\}$ and $\{\bar{x}_p\}$ are convergent.

Let θ_0 and x_0 be their limit, respectively.

Since u is supposed to be $(p \cdot u \cdot b)$, there is $\tau_0 > 0$ such that

$$(4.10) \qquad u(\theta_0, \tau_0, x_0) \in B \subset A.$$

Let $\tau_1 = \tau_0 + \theta_0$ and let p_0 be a positive integer such that $\tau_1 \leq n_0$. We have

$$(4.11) \qquad u(\theta_0, \tau_1 - \theta_0, x_0) \in B$$

$$(4.12) \qquad u(t, \tau, x) \in H_{p_0} \quad \text{for} \quad t \in [0, \omega], \quad \tau \in [0, \tau_1], \quad x \in A.$$

Obviously, we may always assume that B is open, therefore there is $p_1 \geq p_0$ sufficiently large such that

$$(4.13) \qquad u(\bar{\theta}_{p_1}, \tau_1 - \bar{\theta}_{p_1}, \bar{x}_{p_1}) \in B, \qquad \bar{\theta}_{p_1} < \tau_1.$$

Now, it is easy to see that τ_1 is incomparable with $\bar{\tau}_{p_1}$ (which is a contradiction).

Indeed if we assume $\tau_1 \leq \bar{\tau}_{p_1}$ we have $\bar{\theta}_{p_1} < \tau_1 < \bar{\tau}_{p_1}$, so that from (4.9) we derive

$$(4.14) \qquad u(\bar{\theta}_{p_1}, \tau_1 - \bar{\theta}_{p_1}, \bar{x}_{p_1}) \bar{\in} A.$$

But (4.14) contradicts (4.13) (since $B \subset A$).

Finally $\bar{\tau}_{p_1} < \tau_1$ implies $0 < \bar{\tau}_{p_1} - \bar{\theta}_{p_1} < \tau_1$ so that (from 4.12) $u(\bar{\theta}_{p_1}, \bar{\tau}_{p_1} - \bar{\theta}_{p_1}, \bar{x}_{p_1}) \in H_{p_0}$, which contradicts (4.8).

The theorem is thus proved.

Proof of Theorem 4.2. Let u be a ω-periodic process on E^n. Suppose that u is $(p \cdot u \cdot b)$.

We may always assume that B, in Definition 2.5, is open. Let H be an arbitrary compact set in E^n. In the first instance we consider $\sigma = \omega$.

For $y \in H$, there is $t_y > 0$ such that

$$(4.15) \qquad u(\omega, t_y, y) \in B.$$

Since u is continuous and B is open, there is a neighbourhood V_y of y such that

$$(4.16) \qquad u(\omega, t_y, x) \in B \quad \text{for all} \quad x \in V_y.$$

But $\{V_y\}_{y \in H}$ covers H and therefore we can select a finite covering $V_{y_1}, \cdots, V_{y_p}$ of H (where $y_1, \cdots, y_p \in H$) such that

$$(4.17) \qquad u(\omega, t_i, x) \in B \quad \text{for all} \quad x \in V_{y_i}, \qquad i = 1, 2, \cdots, p,$$

where $t_i = t_{y_i}$.

By Theorem 4.1 there is a compact set H′(B) in E^n such that

$$(4.18) \qquad u(\sigma, \tau, y) \in H' \quad \text{for all} \quad \sigma \in R, \quad \tau \in R_+, \quad y \in B.$$

Taking into account (2.3) we derive

$$(4.19) \qquad u(\omega, t - \omega, x) = u(\omega + t_i, t - \omega - t_i, u(\omega, t_i, x)), \qquad t \geq \omega + t_i.$$

Set $T(H) = \max\{t_1, \cdots, t_p\}$. If $x_0 \in H$, there is $i \in \{1, 2, \cdots, p\}$ such that $x_0 \in V_{y_i}$.

Using (4.17), (4.18) and (4.19) we obtain

$$(4.20) \qquad u(\omega, t - \omega, x_0) \in H' \quad \text{for} \quad t \geq \omega + T(H), \text{ i.e.}$$

$$(4.21) \qquad u(\omega, \tau, x) \in H' \quad \text{for all} \quad \tau \geq T(H), \quad x \in H.$$

If $0 \leq \sigma \leq \omega$, we have as usual

$$(4.22) \qquad u(\sigma, t - \sigma, x) = u(\omega, t - \omega, u(\sigma, \omega - \sigma, x)), \qquad t \geq \omega, \quad x \in E^n.$$

There is a compact $\tilde{H}(H)$ in E^n such that

$$(4.23) \qquad u(\sigma, \omega - \sigma, x) \in \tilde{H} \quad \text{for all} \quad x \in H.$$

Using the above result (i.e. (4.21)), there is a positive number T (H) such that

$$(4.24) \qquad u(\omega, \tau, y) \in H'(B) \quad \text{for all} \quad \tau \geq T(\tilde{H}), \quad y \in \tilde{H}.$$

From (4.22), (4.23) and (4.24) it follows

(4.25) $$u(\sigma, t-\sigma, x) \in H' \quad \text{for all} \quad t \geq \omega + T(\tilde{H}), \quad x \in H$$

therefore

(4.26) $$u(\sigma, \tau, x) \in H' \quad \text{for all} \quad \tau \geq \omega + T(\tilde{H}), \quad x \in H,$$

i.e. the process u is $(u \cdot u \cdot b)$ with $B_0 = H'(B)$ and $\tau_0(H) = \omega + T(\tilde{H}(H))$.

The theorem is proved.

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