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Some remarks on structure of polynomially Riesz operators

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Analisi funzionale. — *Some remarks on structure of polynomially Riesz operators.* Nota di GHEORGHE CONSTANTIN, presentata [(*)] dal Socio G. SANSONE.

RIASSUNTO. — L'Autore ottiene nuovi risultati sugli operatori polinomiali di Riesz e alcuni teoremi sulla struttura di questi operatori usando un risultato di T. Andô.

1. Let X be a complex Banach space and $\mathfrak{L}(X)$ the space of bounded linear operators on X. For $T \in \mathfrak{L}(X)$, denote the null manifold $N(T)$ and the range $R(T)$, also the ascent $\alpha(T)$ and the descent $\delta(T)$ as in [13]. We shall write $n(\lambda) = \dim N(T - \lambda I)$, $\alpha(\lambda) = \alpha(T - \lambda I)$ and $\delta(\lambda) = \delta(T - \lambda I)$.

Riesz operators have been introduced by A. F. Ruston [12] and also studied by J. Dieudonné [6], H. Heuser [8], S. Caradus [5], T. T. West [14].

A simple characterization of the set of Riesz operators $\mathfrak{R}$ is given in [5] by: $T \in \mathfrak{R}$ if and only if $\alpha(\lambda)$, $\delta(\lambda)$ and $n(\lambda)$ are finite for all $\lambda \neq 0$.

We say that an operator $T \in \mathfrak{L}(X)$ is polynomially Riesz operator if there exists a non-zero complex polynomial $p(\lambda)$ such that $p(T)$ is a Riesz operator.

The purpose of this Note is to give an extension of a result of [7] for polynomially Riesz operators. Also we obtain some structure theorems of Riesz operators.

2. The non-zero polynomial $p(\lambda)$, of least degree and leading coefficient 1 such that $p(T)$ is a Riesz operator, is called the minimal polynomial of T.

THEOREM 2.1. *Let* T *be a polynomial Riesz operator with minimal polynomial* $p(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_s)^{n_s}$. *Then the Banach space* X *is decomposed into the direct sum* $X = X_1 \oplus \cdots \oplus X_s$ *and* $T = T_1 \oplus \cdots \oplus T_s$ *where* T_i *is the restriction of* T *to* X_i *and* $(T_i - \lambda_i I_i)^{n_i}$ *are all Riesz operators. The spectrum* $\sigma(T)$ *consists of countably many points with* $\{\lambda_1, \cdots, \lambda_s\}$ *as the only possible limit points such that all but possibly* $\{\lambda_1, \cdots, \lambda_s\}$ *are eigenvalues of finite multiplicity. Each point* $\lambda_i \in \{\lambda_1, \cdots, \lambda_s\}$ *is either the limit of eigenvalues of* T *or else* $T_i - \lambda_i I_i$ *is quasi-nilpotent with* X_i *infinite dimensional.*

Proof. First we observe that $\{\lambda_i : p(\lambda_i) = 0\} \subseteq \sigma(T)$. Indeed, in the contrary case we have that $q(T) = (T - \lambda_i I)^{-1} p(T)$ is a Riesz operator since $(T - \lambda_i I)^{-1} \in \mathfrak{L}(X)$ and commutes with $p(T)$, which contradicts the minimality of $p(\lambda)$. Since the structure of the spectrum of Riesz operators is the same as for compact operators, we conclude as in [7] that $\sigma(T)$ consists

(*) Nella seduta del 13 gennaio 1973.

of countable many points with $\lambda_1, \cdots, \lambda_s$ as the only possible limit points. We shall now show that if λ is an isolated point of $\sigma(T)$ such that $p(\lambda) \neq 0$, then λ is an eigenvalue of T of finite multiplicity. Indeed, since λ is isolated then there exists a circle $\gamma(\lambda)$ of centre λ such that λ is the sole point of $\sigma(T)$ inside $\gamma(\lambda)$ and $X = P(\lambda; T) X \oplus (I - P(\lambda; T)) X$ where $P(\lambda; T)$ is the spectral projection associated with the spectral set $\{\lambda\}$, $\sigma(T|_{P(\lambda;T)X}) = \{\lambda\}$ and $P(\lambda; T) X$ is an invariant subspace of T. But $\sigma(p(T|_{P(\lambda;T)X})) = \{p(\lambda)\}$ and $p(\lambda) \neq 0$ so that $p(T|_{P(\lambda;T)X})$ is an invertible Riesz operator which implies that $\dim(P(\lambda; T) X) < \infty$ and $P(\lambda; T) X = \{x : (T - \lambda I)^k x \neq 0\}$ for an integer k.

Let $\lambda \in \sigma(T)$, $p(\lambda) = 0$ and such that λ is an isolated point in $\sigma(T)$. As above, we obtain that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, $(T_2 - \lambda I_2)^{-1} \in \mathfrak{L}(X_2)$ and commutes with $p(T_2)$. Then $(T_2 - \lambda I_2)^{-1} p(T_2)$ is a Riesz operator and therefore $\dim(X_1) = \infty$ since in the contrary case the polynomial $p(\lambda)$ is not minimal. It follows that $T_1 - \lambda I_1$ is quasi-nilpotent and λ has infinite multiplicity.

Hence to complete the proof of this theorem we need only show that $(T_i - \lambda_i I_i)^{n_i}$, $i = 1, 2, \cdots, s$ are Riesz operators, since the requested decompositions are obtained by induction. For this we observe that $T_i - \lambda_j I_i$ is invertible for each $i \neq j$ and $(T_i - \lambda_i I_i)^{n_i} = \prod_{j \neq i} (T_i - \lambda_j I_i)^{-n_j} p(T_i)$.

In what follows, as in [4], H denotes an infinite dimensional Hilbert space, $\omega(T)$ the Weyl spectrum of T and T is said to satisfy condition (α') if every direct summand of T satisfies (G_1) (i.e., $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$).

COROLLARY 2.1. *Let* $X = H$, T *a polynomially Riesz operator which satisfies* (α'). *Then* T *is a normal operator and* $\omega(T)$ *is a finite set.*

Proof. From the Theorem 2.1, $\sigma(T)$ is countable and then by [3, Theorem 1] T is a diagonal normal operator. Since for normal operators the spectral mapping theorem for $\omega(T)$ holds we have $p(\omega(T)) = \omega(p(T)) = \{0\}$ and therefore $\omega(T)$ is a finite set.

By [3, Theorem 3] T is in fact a polynomially compact operator.

It is known [4] that $\omega(T) = \{0\}$ for any compact operator T and the converse is false; in [4] are given sufficient conditions for compactness of T. We will show that the condition (ii) of Theorems 6.8 and 7.1 of [4] is superfluous.

We recall that an operator T is called convexoid if $\operatorname{conv} \sigma(T) = \overline{W(T)}$ where $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ is the numerical range of T.

THEOREM 2.2. *If* $\omega(T) = \{0\}$ *and the restriction of* T *to any invariant subspace is convexoid, then* T *is compact and normal operator.*

Proof. A characterization of Riesz operators due by A. F. Ruston [12] asserts that $T \in \mathfrak{R}$ if and only if $\sigma(\tilde{T}) = \{0\}$ where $\tilde{T}$ is the image of T in the Calkin algebra $\mathfrak{L}(H)/\mathfrak{K}(H)$ [$\mathfrak{K}(H)$ is the ideal of all compact operators].

Since $\sigma(\bar{T}) \subseteq \omega(T)$ it follows that $\sigma(\bar{T}) = \{0\}$ and thus T is a Riesz operator. On the other hand $\sigma(T)$ has only one limit point and by [2, Lemma 4], T is normal. But a normal Riesz operator is compact.

COROLLARY 2.2. *If* $T = C + Q$ *with* $C =$ *compact and* $\sigma(Q) = \{0\}$ *and the restriction of* T *to any invariant subspace is convexoid, then* T *is compact and normal operator.*

COROLLARY 2.3. *If* $\omega(T) = \{\lambda\}$ *and the restriction of* T *to every invariant subspace is convexoid, then* $T = \lambda I + C$ *with* C *compact and normal.*

THEOREM 2.3. *If* $\omega(T) = \{\lambda\}$, $\lambda \neq 0$ *and the restriction of* T *to every invariant subspace is convexoid then* T *is a normal noncommutator.*

3. An operator $T \in \mathfrak{L}(H)$ is of class (N) if $\| T^2 x \| \geq \| Tx \|^2$ for all $x \in H$, $\| x \| = 1$ [10]. In [1] is given a characterization of operators of class (N) which suggests a generalization of some structure theorems.

THEOREM 3.1. *If* T *is of class* (N) *then its image* $\bar{T}$ *in the Calkin algebra is also of class* (N).

Proof. From Andô's theorem [1] we have that T is of class (N) if and only if

$$T^{*2} T^2 - 2\lambda T^* T + \lambda^2 I \geq 0$$

for all $\lambda > 0$. Considering the image of T in the Calkin algebra we obtain

$$\bar{T}^{*2} \bar{T}^2 - 2\lambda \bar{T}^* \bar{T} + \lambda^2 I \geq 0$$

i.e., $\bar{T}$ is of class (N).

THEOREM 3.2. *If* T *is an operator of class* (N) *and*

$$T^{*p_1} T^{q_1} \cdots T^{*p_n} T^{q_n} = C$$

where $p_1, q_1, \cdots, p_n, q_n$ *are positive integers and* C *is a compact or Riesz operator then* T *is a normal operator.*

Proof. If we consider the image in the Calkin algebra we obtain

$$\bar{T}^{*p_1} \bar{T}^{q_1} \cdots \bar{T}^{*p_n} \bar{T}^{q_n} = 0$$

from which it follows that $\bar{T} = 0$, i.e., T is compact and by [10, Theorem 2.2] we conclude that T is normal.

THEOREM 3.3. *If* T *is an operator of class* (N) *and*

$$\sum_{k=0}^{\infty} a_k T^{*k} T^k = C$$

where C *is a compact or Riesz operator and* $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$ *is an entire function nonvanishing on real positive numbers then* T *is normal.*

Proof. The operator T has the property that

$$\sum_{k=0}^{\infty} a_k \bar{T}^{*k} \bar{T}^k = 0$$

and since $\bar{T}$ is normaloid we obtain that $\bar{T} = 0$ and the result follows by [9, Theorem 3.1].

In [6] is given a characterization of quasi-hermitian Riesz operators (an operator T is quasi-hermitian if there exists a hermitian operator $S > 0$ such that $ST = T^* S$).

THEOREM 3.4. *Let* T *be a quasi-hermitian operator for which the operator* S *is not compact. If* T *is a Riesz operator then* T *is compact.*

Proof. It is known that every Riesz operator T has the form $T = C + Q$ where C is compact and Q quasi-nilpotent. If we consider the image in the Calkin algebra we obtain that $\bar{S} \neq 0$ and $\bar{T} = \bar{Q}$. Let $\bar{Q} \neq 0$, then since $\sigma(\bar{Q}) = \{0\}$ and the operator $\bar{Q}$ is quasi-hermitian it follows that $\bar{Q} = 0$, which is a contradiction.

Remark. The condition of quasi-hermiticity for a compact operator is more simple than for other operators.

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