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Fiber bundle with involution and characteristic classes

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Topologia algebrica. — *Fiber bundle with involution and characteristic classes* (*). Nota di NICOLAE TELEMAN (**), presentata (***) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — In questo lavoro costruisco e studio le proprietà di un sistema di classi caratteristiche t_i per fibrati localmente banali « muniti di una involuzione ». Le classi t_i generalizzano le classi di Stiefel-Whitney. Il procedimento costruttivo assomiglia alla costruzione delle operazioni coomologiche di Steenrod.

1. INTRODUCTION

It is known [5] that the Stiefel-Whitney characteristic classes of a real vector bundle E can be defined by the formula $w_i = \varphi^{-1} S_q^i U$, where φ is the Thom isomorphism in cohomology, S_q^i is the i-th Steenrod squaring operation, and U the Thom class of the vector bundle.

It is also known [1] that the Stiefel-Whitney characteristic classes can be defined by using the covering map $S(E \oplus 1) \to P(E \oplus 1)$, where S(E), resp. P(E) denotes the associated sphere, resp. projective bundle with E.

If X is a topological space, the involution $T : X \times X \to X \times X$, $T : (x_1, x_2) \mapsto (x_2, x_1)$ is used for the definition of the Steenrod squaring operations. The definition of the covering map $S(E \oplus 1) \to P(E \oplus 1)$ requests also an involution, the antipodal involution A, defined on each sphere of the bundle $S(E \oplus 1)$. Hence, the bide constructions of the Stiefel-Whitney characteristic classes involve an involution; while the involution T is " external " (the involution T is not defined on X, but on $X \times X$), the involution A is " internal " (the involution A is defined on the space $S(E \oplus 1)$).

We consider fiber bundles ξ with fiber F which is $(n - 1, R)$-simple, (R being a commutative ring with 1), i.e. $H_i(F, R) = 0$ for $1 \leq i \leq n - 1$. In the bundle ξ we consider an arbitrary fiber preserving involution.

In these hypotheses we construct a system of characteristic classes $t_i(\xi)$ which generalizes the Stiefel-Whitney characteristic classes. In particular, for $F = S^n$, and $R = Z_2$ our classes satisfy all the axioms of Stiefel-Whitney characteristic classes less one of them; our characteristic class $t_0(\xi)$ is not necessarily 1; for example, if the base is connected by arcs and in the total space of ξ there exists a fixed point at the involution, then $t_0(\xi) = 0$. If there exists a continuous section of fixed points then all $t_i(\xi)$ vanish.

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We define also a cohomological invariant of involutions, called the " index of the involution ".

Connected problems are studied by I. M. James and D. W. Anderson in [2], [3], [4].

I present here a summary of the results which I have obtained. I shall give an extensive exposition in a subsequent paper.

2. FIBER BUNDLES WITH INVOLUTION

2.1. DEFINITION. A " fiber bundle with involution " is a quintuple $\xi = (E, \pi, B, F, A)$, where $E \xrightarrow{\pi} B$ is a local trivial fiber bundle with fiber F; and $A : E \to E$ is a continuous, involutive, fiber-preserving map ($\pi A = \pi$, $A^2 = 1$).

We suppose that any local trivial fiber bundle with fiber F over an arbitrary simplex is a product bundle.

2.2. DEFINITION. If $\xi_i = (E_i, \pi_i, B, F_i, A_i)$, $i = 1, 2$, are fiber bundles with involution, then these are " equivalent " if and only if there exists a homeomorphism $f : E_1 \to E_2$ such that $\pi_2 f = \pi_1$, $A_2 f = f A_1$, and f maps E_1 homeomorphically on E_2.

Let $\mathfrak{B}(B, F_1, Z_2)$, denote the set of equivalence classes of fiber bundles with involution with fibers F_1.

2.3. DEFINITION. If $\xi = (E, \pi, B, F, A) \in \mathfrak{B}(B, F, Z_2)$ and $f : B' \to B$ is a continuous map, then in the pull-back $f^* E$

$$\begin{array}{ccc} f^* E & \xrightarrow{\bar{f}} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

there exists a unique involution $f^* A$ such that $\bar{f} \circ (f^* A) = A \bar{f}$; hence there exists a well defined $f^* \xi = (f^* E, f^* \pi, B', F, f^* A) \in \mathfrak{B}(B', F, Z_2)$.

2.4 DEFINITION. If $\xi = (E, \pi, B, F, A) \in \mathfrak{B}(B, F, Z_2)$, let $C\xi \in \mathfrak{B}(B, CF, Z_2)$ denote the associated cone bundle with the involution $\hat{A}$ such defined: $\hat{A}(f, t) = (Af, t)$, $(f, t) \in F_x \times I / F_x \times \{0\}$.

2.5. DEFINITION. If $\xi = (E, \pi, B, F, A) \in \mathfrak{B}(B, F, Z_2)$ let $\Sigma\xi = (\hat{\hat{E}}, \hat{\hat{\pi}}, B, \Sigma F, \hat{\hat{A}}) \in \mathfrak{B}(B, \Sigma F, Z_2)$ denote the bundle with involution which has fiber $E_x = \Sigma E_x$, $x \in B$; the involution $\hat{\hat{A}}$ is defined as follows: in $\Sigma F = C_{-1} F \sqcup_F C_{+1} F$ ($C_{\pm 1} F$ denoting the cone over F), if $e = (f, t) \in C_{\pm 1} F$, $0 \leq t \leq 1$, $f \in F$, then $\hat{\hat{A}} e = (Af, t) \in C_{\mp 1} F$.

3. THE EILENBERG SUBCOMPLEX

Let R be a fixed commutative ring with 1. Let $C_*(—, R)$ denote the singular chain complex (with coefficients in R) functor; let $\partial : C_*(—, R) \circlearrowleft$ denote the boundary operator.

3.1. DEFINITION. If F is a topological space, we say that F is (n, R)-simple if $H_i(F, R) = 0$ for $0 < i \leq n$.

3.2. DEFINITION. Let F be a $(n-1, R)$-simple space and $\xi \in \mathcal{B}(B, F, Z_2)$. By definition, the "Eileberg relative subcomplex" of $C\xi$ (see 2.4) is $\mathcal{E}_*^{(n-1)}(C\xi, R) \subset C_*(C\xi, R)$ such defined: $\mathcal{E}_k^{(n-1)}(C\xi, R)$ is free over R; the generators of $\mathcal{E}_k^{(n-1)}(C\xi, R)$ are that and only that singular k-simplexes $\sigma : \Delta^k \to C\xi$ for which $\sigma(\Delta^k)^{(n)} \subset C\xi_0 \cdot ((\Delta^k)^{(n)}$ denotes the n-dimensional skeleton of Δ^k).

3.3. PROPOSITION. *If* $\xi \in \mathcal{B}(B, F, Z_2)$ *and* F *is* $(n-1, R)$-simple, *then the inclusion* $\mathcal{E}_*^{(n-1)}(C\xi, R) \xrightarrow{i} C_*(C\xi, R)$ *is a chain homotopy equivalence.*

4. HOMOLOGICAL LOCAL SYSTEMS IN FIBER BUNDLE WITH INVOLUTION

Let be $\xi = (E, \pi, B, F, A) \in \mathcal{B}(B, F, Z_2)$, and R a fixed commutative ring with 1; let n be a fixed natural number.

For any point $b \in B$ we consider the homology R-module $H_n(E_b, R)$. Let $\rho : [0, 1] \to B$ be a path. In the total space $\rho^* E$ of $\rho^* \xi$ we have the natural inclusions

$$E_{\rho(0)} = (\rho^* E)_0 \xrightarrow{i_0} \rho^* E \xleftarrow{i_1} (\rho^* E)_1 = E_{\rho(1)}.$$

As the fiber bundle (without involution) $\rho^* E \to I$ is equivalent to a product bundle, i_0 and i_1 are homotopic equivalences; hence, i_0 and i_1 induce isomorphisms in homology, and in consequence the well determinated isomorphism

$$|\rho| = (i_1)_*^{-1} \circ (i_0)_* \quad : \quad H_n(E_{\rho(0)}, R) \to H_n(E_{\rho(1)}, R).$$

The isomorphism $|\rho|$ depends only on the homotopy class of ρ. Really, if $\rho_\tau : I \to B$ is a homotopy with fixed ends, then a similar argument applicated to the fiber bundle $\tilde{\rho}^* E \xrightarrow{\tilde{\rho}^* \pi} I \times I$, where $\tilde{\rho} : I \times I \to B$ is $\tilde{\rho}(\tau, t) = \rho_\tau(t)$, conducts to the assertion.

Now we take in consideration the involution A in connection with $|\rho|$.

If we denote in general $A_b = A|_{E_b}$, we have the commutative diagram for the upper path:

$$\begin{array}{ccccccccc} E_{\rho(0)} & = & (\rho^* E)_0 & \xrightarrow{i_0} & \rho^* E & \xleftarrow{i_1} & (\rho^* E)_1 & = & E_{\rho(1)} \\ \downarrow A_{\rho(0)} & & \downarrow (\rho^* A)_0 & & \downarrow \rho^* A & & \downarrow (\rho^* A)_1 & & \downarrow A_{\rho(1)} \\ E_{\rho(0)} & = & (\rho^* E)_0 & \xrightarrow[i_0]{} & \rho^* E & \xleftarrow[i_1]{} & (\rho^* E)_0 & = & E_{\rho(0)} \end{array}$$

from which derives the commutativity in homology

$$(1) \qquad |\rho|\,(A_{\rho(0)})_* = (A_{\rho(1)})_*\,|\rho|\,.$$

If $b \in B$, let be

$$H_b = H_n(E_b, R) \quad , \quad H_b^{\pm} = \frac{H_b}{(1 \pm (A_b)_*)\, H_b}\,.$$

The relation (1) shows that $|\rho|$ induces two isomorphisms:

$$|\rho|^{\pm} : H^{\pm}_{\rho(0)} \to H^{\pm}_{\rho(1)}\,,$$

which depend only on the homotopy class of the path ρ.

Let $\mathcal{H}^{\pm}(\xi, R)$, resp. $\mathcal{H}(\xi, R)$ denote the local systems $(H_b^{\pm}, |\rho|^{\pm})$, resp. $(H_b, |\rho|)$. These two local systems are called the "homological local systems of the fiber bundle with involution ξ in dimension n".

5. On the Thom isomorphism

5.1. Theorem. *Let* $E \xrightarrow{\pi} B$ *be a local trivial fiber bundle with fiber* F $(n-1, R)$*-simple (see 3.1), and* $S = (S_b, |\rho|)$ *a* R*-local system over* B.

Then $\pi^* : \mathcal{H}^r(B, S) \to \mathcal{H}^r(E, \pi^* S)$ is an isomorphism for $0 \le r \le n-1$ and is a monomorphism for $r = n$.

6. Characteristic classes of fiber bundles with involution

6.1. Theorem. *Let be* $\xi = (E, \pi, B, F, A) \in \mathcal{B}(B, F, Z_2)$ *and* F *let be* $(n-1, R)$*-simple and connected by arcs. Then there exist the local* R*-homomorphisms*:

i) $k_p^{(r)} : C_p(E, R) \to C_{p+r}(E, R)\,, 0 \le p + r \le n\,, k_p^{(0)} = \text{id.},$

such that:

$$(1 + (-1)^r A)\, k_p^{(r-1)} = \partial_{p+r}\, k_p^{(r)} + (-1)^{r+1}\, k_{p-1}^{(r)}\, \partial_p\,,$$

ii) *if* $k_p^{(r)}, \tilde{k}_p^{(r)}$ *are two such systems of local homomorphisms which satisfy* i), *then there exists the system of local* R*-homomorphisms*:

$$\varphi_p^{(r)} : C_p(E, R) \to C_{p+r+1}(E, R)\,, \qquad p + r + 1 \le n,$$

such that, if we denote $K_p^{(r)} = k_p^{(r)} - \tilde{k}_p^{(r)}$, *we have*:

$$K_p^{(r)} = (1 + (-1)^r A)\, \varphi_p^{(r-1)} + \partial \varphi_p^{(r)} + (-1)^r\, \varphi_{p-1}^{(r)}\, \partial\,.$$

iii) *There exist the local* R*-homomorphisms*

$$\mu_{n-r}^{(r)} : C_{n-r}(E, R) \to C_n(E, R)$$

such that:

$$K_{n-r}^{(r)} = (1 + (-1)^r A)\, \varphi_{n-r}^{(r-1)} + (-1)^r\, \varphi_{n-r-1}^{(r)}\, \partial + \mu_{n-r}^{(r)}\,, \qquad \partial \mu_{n-r}^{(r)} = 0\,.$$

6.2. We know from the preceding considerations that:

(1) $$(\hat{\omega}_r(\xi\,,k_p^{(r)}))(\sigma) = ((1+(-1)^r A)\,k_{n-r+1}^{(r-1)} - (-1)^{r+1}\,k_{n-r}^{(r)}\,\partial)(\sigma)$$

is a cycle; let be, for an arbitrary singular simplex $\sigma \in \nabla_{n-r+1}(E)$

$$(\omega_r(\xi\,,k_p^{(r)}))(\sigma) = [(\hat{\omega}_r(\xi\,,k_p^{(r)}))(\sigma)]$$

where $[\gamma]$ denotes the homology class of the cycle γ. Therefore $\omega_r(\xi\,,k_p^{(r)}) \in C^{n-r+1}(E\,,\pi^*\mathcal{H}_n(\xi\,,R))$ = the R-module of $(n-r+1)$-singular cochains with coefficients in the local system $\pi^*\mathcal{H}_n(\xi\,,R)$.

The coboundary of $\omega_r(\xi\,,k_p^{(r)})$ is

$$(dw_r(\xi\,,k_p^{(r)}))(\sigma) = ((-1)^{r-1} - A_*)\,(\omega_{r-1}(\xi\,,k_p^{(r)}))(\sigma)\,.$$

6.3. *Notation*. Let $\chi^{\pm}$ denote the canonical epimorphisms of local systems:

$$\chi^{\pm} : \mathcal{H}_n(\xi\,,R) \to \mathcal{H}_n^{\pm}(\xi\,,R)\,;$$

in consequence we have the exact sequences:

(1) $$0 \to (1 \pm A_*)\,\mathcal{H}_n(\xi\,,R) \hookrightarrow \mathcal{H}_n(\xi\,,R) \xrightarrow{\chi^{\pm}} \mathcal{H}_n^{\pm}(\xi\,,R) \to 0\,.$$

We obtain the following:

6.4. THEOREM. *If* $\xi = (E\,,\pi\,,B\,,F,Z_2) \in \mathcal{B}(B\,,F\,,Z_2)$, *the fiber* F *being* $(n-1\,,R)$-*simple and connected by arcs, then for any* $0 \leq r \leq n$ *we can define the cochain* $\omega_r(\xi\,,k_p^{(r)}) \in C^{n-r+1}(E\,,\pi^*\mathcal{H}_n(\xi\,,R))$, *where* $k_p^{(r)}$ *are defined in the Theorem 6.1* i). *The coboundary of* $\omega_r(\xi\,,k_p^{(r)})$ *is*:

$$d\omega_r(\xi\,,k_p^{(r)}) = ((-1)^{r-1} - A_*)\,\omega_{r-1}(\xi\,,k_p^{(r)})\,,$$

and, in consequence

$$\tilde{\omega}_r(\xi\,,k_p^{(r)}) = \chi^{\varepsilon_r}\,\omega_r(\xi\,,k_p^{(r)}) \quad , \quad \varepsilon_r = \mathrm{sign}\,(-1)^r$$

is a cocycle.

6.5. THEOREM. *The cohomology class* $[\tilde{\omega}_r(\xi\,,k_p^{(r)})] \in \mathcal{H}^{n-r+1}(E\,,\mathcal{H}_n^{\varepsilon_r}(\xi,R))$ *is independent of the choice of the local homomorphisms* $k_p^{(r)}$ *from Theorem 6.1*.

6.6. THEOREM. *The cohomology class* $[\tilde{\omega}_r(\xi\,,k_p^{(r)})]$ *defined in the Theorem 6.4. is a basic class, i.e.*

$$[\tilde{\omega}_r(\xi\,,k_p^{(r)})] \in \pi^*\mathcal{H}^{n-r+1}(B\,,\mathcal{H}_n^{\varepsilon_r}(\xi\,,R)).$$

6.7. DEFINITION. If $\xi \in \mathcal{B}(B\,,F\,,Z_2)$ and if ΣF is $(n-1\,,R)$-simple, we shall write $\xi \in \mathcal{B}_R^n(B\,,F\,,Z_2)$. The suspension ΣF is connected by arcs, and in $\hat{\hat{\Sigma\xi}}$ there exist two canonical sections: the zero sections $s_{\pm 1} : B \to E_{\pm 1} \hookrightarrow \hat{\hat{E}}$, where $E_{\pm 1} = C_{\pm}\,\xi$.

6.8. DEFINITION. If $\xi \in \mathcal{B}^n(B, F, Z_2)$, then the R-characteristic classes $t_i(\xi)$ of $\xi = (E, \pi, B, F, A)$ are

$$t_i(\xi) = s_{+1}^{*}[\tilde{\omega}_{n-i+1}(\Sigma\xi, k_p^{(r)})] \in$$

$$\in \mathcal{H}^i(B, \mathcal{H}_n^{\varepsilon_{n-i+1}}(\Sigma\xi, R)) \quad , \quad 0 \le i \le n \quad , \quad \varepsilon_r = \operatorname{sign}(-1)^r .$$

6.9. Let be $\xi \in \mathcal{B}^n(B, F, Z_2)$; then $\Sigma\xi \in \mathcal{B}_R^{n+1}(B, F, Z_2)$. For the calculation of $t_i(\xi)$, resp. $t_i(\Sigma\xi)$, we must consider the local systems:

$$\mathcal{H}_{n+1}^{\pm}(\Sigma\xi, R), \qquad \text{resp. } \mathcal{H}_{n+1}^{\pm}(\Sigma^2\xi, R).$$

We remark that by the suspension isomorphism theorem in homology, which is natural, we have the equivalence of local systems

$$\mathcal{H}_{n+1}^{\pm}(\Sigma^2\xi, R) \xrightarrow{\Sigma} \mathcal{H}^{\mp}(\Sigma\xi, R).$$

6.10. THEOREM. *The characteristic classes t_i have the properties*:

(0) *for* $\xi \in \mathcal{B}_R^n(B, F, Z_2)$, $t_i(\xi) \in \mathcal{H}^i(B, \mathcal{H}_n^{\varepsilon_{n-i+1}}(\xi, R))$, $0 \le i \le n$,

(i) *if* $\xi \in \mathcal{B}_R^n(B, F, Z_2)$ *and* $f : B_1 \to B$ *is a continuous map, then*

$$t_i(f^*\xi) = f^*(t_i(\xi)).$$

(ii) *if* $\xi \in \mathcal{B}_R^n(B, F, Z_2)$, *then* $\Sigma\xi \in \mathcal{B}_R^{n+1}(B, F, Z_2)$ *and, by respect the equivalence of the local systems defined by the suspension isomorphism*

$$t_i(\Sigma\xi) = (-1)^{n-i} A_* t_i(\xi), \qquad 0 \le i \le n,$$
$$t_{n+1}(\Sigma\xi) = 0;$$

(iii) (*the "Whitney duality formula"*) *if* $\xi_1 \in \mathcal{B}_R^m(B, F_1, Z_2)$, $\xi_2 \in \mathcal{B}_R^n(B, F_2, Z_2)$ *and if we denote by resp.* χ_1, χ_2, χ *the corresponding epimorphisms from 7.3 for resp.* $\xi_1, \xi_2, \xi_1 \oplus \xi_2$, *then there exist the cocycles*

$$\alpha_p \in C^p(B, \mathcal{H}_m(\xi_1, R)), \qquad 0 \le p \le m,$$
$$\beta_q \in C^q(B, \mathcal{H}_n(\xi_2, R)) \qquad 0 \le q \le n,$$

such that

$$t_s(\xi_1 \oplus \xi_2) = \chi\left[\sum_{p+q=S} (-1)^{\varepsilon(p,q)} A_{1*}^{n-q} \alpha_p \oplus A_{2*} \beta_q\right]$$

and

$$\chi_1 \alpha_p = t_p(\xi_1)$$
$$\chi_2 \beta_q = t_q(\xi_2).$$

where

$$\varepsilon(p, q) = p(n - q + 1) + m + n + 1.$$

(iv) *if in* $\xi \in \mathcal{B}_R^n(B, F, Z_2)$ *there exists a continuous section of fixed points for the involution in* ξ, *then*:

$$t_i(\xi) = 0 \qquad \text{for } 0 \le i \le n.$$

7. THE INDEX OF AN INVOLUTION

If F is a topological space with involution such that ΣF is $(n - 1, R)$-simple, then we can consider F as a fiber space with involution over a point. Hence $t_0(F) \in \dfrac{H_n(F, R)}{(1 + (-1)^{n+1} A_*) H_n(F, R)}$.

7.1. DEFINITION. If F is a topological space with involution such that ΣF is $(n - 1, R)$-simple, then $t_0(F)$ will be called the index of the involution, and will be denoted $I_R^n(F, A)$.

7.2. THEOREM. *The index of the involution has the properties*:

(i) *if* $A_t : F \to F$ *is a continuous deformation of the involution* A_0 *in the involution* A_1, *then*

$$I_R^n(F, A_0) = I_R^n(F, A_1).$$

(ii) *If* F_1 *resp.* F_2 *are two topological spaces with involutions* A_1, *resp.* A_2, *and* ΣF_1, *resp.* ΣF_2 *are* $(m - 1, R)$-*simple, resp.* $(n - 1, R)$-*simple, then* $I_R^{m+n+1}(F_1 \oplus F_2, A_1 \oplus A_2) = (-1)^{m+n+1} A_{1*}^n I_R^m(F_2, A_1) \cdot A_{2*} I_R^n(F_2, A_2)$.

(iii) *If* A *is an involution in* F, *and if* A *has at least a fixed point, then* $I_R^n(F, A) = 0$.

7.3. COROLLARY. *If* $\xi \in \mathfrak{B}_R^n(B, F, Z_2)$ *and if* B *is connected by arcs, if there exists at least a fixed point in the total space of* ξ, *then* $t_0(\xi) = 0$.

8. CHARACTERISTIC CLASSES OF INVOLUTIONS IN SPHERE BUNDLES

In this section we particularize the coefficients to Z_2 and we consider only spherical fibers. Then the Theorem 6.10 becomes:

8.1. THEOREM. *The characteristic classes of sphere bundles with involution have the properties*:

(o) *for* $\xi \in \mathfrak{B}_{Z_2}^n(B, S^{n-1}, Z_2)$, $t_i(\xi) \in H^i(B, Z_2)$, $\quad 0 \leq i \leq n$,

(i) *if* $\xi \in \mathfrak{B}_{Z_2}^n(B, F, Z_2)$ *and* $f : B_1 \to B$ *is a continuous map, then*

$$t_i(f^* \xi) = f^*(t_i(\xi)),$$

(ii) *if* $\xi \in \mathfrak{B}_{Z_2}^n(B, S^{n-1}, Z_2)$, *then*

$$t_i(\Sigma\xi) = t_i(\xi) \qquad 0 \leq i \leq n$$

$$t_{n+1}(\Sigma\xi) = 0,$$

(iii) *if* $\xi_1 \in \mathfrak{B}_{Z_2}^m(B, S^{m-1}, Z_2)$, $\xi_2 \in \mathfrak{B}_{Z_2}^n(B, S^{n-1}, Z_2)$,

then

$$t_i(\xi_1 \oplus \xi_2) = \sum_{p+q=i} t_p(\xi_1)\, t_q(\xi_2),$$

(iv) *if in* $\xi \in \mathcal{B}^n_{Z_2}(B, S^{n-1}, Z_2)$ *there exists a continuous section of fixed points, then*

$$t_i(\xi) = 0, \qquad 0 \leq i \leq n,$$

(v) *if* $\xi \in \mathcal{B}^n_{Z_2}(B, S^{n-1}, Z_2)$, *then*

$$t_n(\xi) = w_n(\xi)$$

8.2. *Remark.* The classes t_i satisfy all Stiefel-Whitney axioms less one of them: $t_0(\xi)$ can be 0, while $w_0(\xi)$ is ever 1.

8.3. *Remark.* For the classes t_i the relation $t_i(\xi) = w_i(\xi) \cdot t_0(\xi)$ is generally false.

8.4. THEOREM. *If* $\xi \in \mathcal{B}^n_{Z_2}(B, S^{n-1}, Z_2)$ *and if* ξ *is in addition an Euclidean sphere bundle (associated with a real vector bundle) provided with the antipodal involution, then*

$$t_i(\xi) = w_i(\xi).$$

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