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A Comparison Theorem for Focal Points of Nonselfadjoint Differential Equations of Even Order

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A Comparison Theorem for Focal Points of Nonselfadjoint Differential Equations of Even Order* (*). Nota di KURT KREITH, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — È stabilito un teorema di confronto per una classe di punti focali di una non autoaggiunta equazione differenziale di ordine pari. È dato un semplice esempio secondo il quale il detto criterio non può essere esteso a punti coniugati.

1. INTRODUCTION

Oscillation properties of solutions of differential equations are intimately related to an ordering of associated differential operators. The classical example of this is given by the selfadjoint Sturm–Liouville equations

$$lu \equiv -(p_1(x)\,u')' + p_0(x)\,u = 0\,, \tag{1.1}$$

$$Lv \equiv -(P_1(x)\,v')' + P_0(x)\,v = 0\,. \tag{1.2}$$

If there exists a nontrivial solution $u(x)$ of (1.1) with zeros at $x = \alpha$ and $x = \beta$, and if the coefficients of (1.1) and (1.2) satisfy

$$\begin{aligned} &p_1(x) \geq P_1(x) > 0 \\ &p_0(x) \geq P_0(x) \end{aligned} \tag{1.3}$$

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for $\alpha \leq x \leq \beta$, then the Sturm–Picone theorem asserts that every solution of (1.2) has a zero in $[\alpha, \beta)$. The condition (1.3) can be interpreted as " $L \leq l$ ". Indeed, in terms of the inner product on $L^2[\alpha, \beta]$, the inequalities (1.3) lead to the inequality

$$(L\varphi, \varphi) \leq (l\varphi, \varphi) \tag{1.4}$$

for all sufficiently regular $\varphi(x)$ with zeros at $x = \alpha$ and $x = \beta$, and this property can be used to establish Sturmian comparison theorems for selfadjoint equations [1].

These considerations allow generalizations in two separate directions. For the nonselfadjoint equations

$$lu \equiv -(p_1(x)\,u')' + q_0(x)\,u' + p_0(x)\,u = 0\,, \tag{1.4}$$

$$Lv \equiv -(P_1(x)\,v')' + Q_0(x)\,v' + P_0(x)\,v = 0\,, \tag{1.5}$$

there is a direct generalization of the Sturm–Picone theorem [2], [3], [4]. Some appropriate interpretations of " $L \leq l$ " are now that

$$\begin{aligned} &p_1(x) \geq P_1(x) > 0 \\ &p_0(x) \geq P_0(x) + \frac{Q_0^2(x)}{4P_1(x)} - \frac{1}{2}\frac{d}{dx}[Q_0(x) - q_0(x)] \end{aligned} \tag{1.6}$$

or that

$$\begin{aligned} &p_1(x) > P_1(x) > 0 \\ &p_0(x) \geq P_0(x) + \frac{Q_0^2(x)}{4P_1(x)} + \frac{(q_0(x) - Q_0(x))^2}{p_1(x) - P_1(x)}\,. \end{aligned} \tag{1.7}$$

Another generalization is to selfadjoint equations of arbitrary even order

$$lu \equiv \sum_{k=0}^{n} (-1)^k (p_k(x)\,u^{(k)})^{(k)} = 0 \tag{1.8}$$

$$Lv \equiv \sum_{k=0}^{n} (-1)^k (P_k(x)\,v^{(k)})^{(k)} = 0 \tag{1.9}$$

where oscillatory behavior is determined by location of *conjugate points*.

It can readily be shown [5] that if $\mu_1(\alpha)$ is the smallest $\beta > \alpha$ such that a nontrivial solution of (1.8) has n-th order zeros at $x = \alpha$ and $x = \beta$, and if $L \leq l$ in the sense that

$$\begin{aligned} &p_n(x) \geq P_n(x) > 0 \qquad \text{on} \quad [\alpha, \mu_1(\alpha)] \\ &p_k(x) \geq P_k(x) \qquad\quad\;\; \text{on} \quad [\alpha, \mu_1(\alpha)] \qquad \text{for} \quad k = 0, \cdots, n-1, \end{aligned} \tag{1.10}$$

then (1.9) has a nontrivial solution with two n-th order zeros in $[\alpha, \mu_1(\alpha)]$.

The question naturally arises as to whether there is an appropriate interpretation of $L \leq l$ in case these operators are nonselfadjoint *and* of arbitrary even order. Specifically, if

$$(1.11)\qquad lu \equiv \sum_{k=0}^{n} (-1)^k (p_k(x)\, u^{(k)})^{(k)} + \sum_{k=0}^{n-1} (-1)^k (q_k(x)\, u^{(k+1)})^{(k)} = 0$$

$$(1.12)\qquad Lv \equiv \sum_{k=0}^{n} (-1)^k (P_k(x)\, v^{(k)})^{(k)} + \sum_{k=0}^{n-1} (-1)^k (Q_k(x)\, v^{(k+1)})^{(k)} = 0$$

is there an appropriate definition of $L \leq l$ such that solutions of (1.12) oscillate at least as fast as solutions of (1.11)? Affirmative answers have been given in terms of conjugate points [6], [7], [8] when l is selfadjoint (i.e. $q_k(x) \equiv 0$ for $k = 0, \cdots, n-1$), but no such result seems to be known when (1.11) is nonselfadjoint.

The purpose of this note is to show that there is an appropriate definition of $L \leq l$ when oscillation is measured in terms of *focal points* rather than conjugate points. A simple example will be given to show that this ordering of operators does *not* have the desired implication when oscillation is measured in terms of conjugate points.

2. FORMULATION OF THE PROBLEM

The differential equation (1.11) will be written recursively

$$u_{(n)} = p_n(x)\, u^{(n)}$$

$$u_{(n-k)} = -u'_{(n-k+1)} + q_{n-k}(x)\, u^{(n-k+1)} + p_{n-k}(x)\, u^{(n-k)}, \qquad k = 1, 2, \cdots, n-1$$

$$u'_{(1)} = q_0(x)\, u' + p_0(x)\, u,$$

this form requiring only continuity (or integrability) of the coefficients on the interval under consideration. This recursive system can in turn be expressed as a vector system of the form

$$(2.1)\qquad \begin{aligned} \boldsymbol{u}' &= a(x)\, \boldsymbol{u} + b(x)\, \boldsymbol{w} \\ \boldsymbol{w}' &= c(x)\, \boldsymbol{u} + d(x)\, \boldsymbol{w} \end{aligned}$$

where $\boldsymbol{u}(x)$ and $\boldsymbol{w}(x)$ are column n vectors defined by

$$\boldsymbol{u}(x) = \mathrm{col}\,(u(x), u'(x), \cdots, u^{(n-1)}(x)) \quad , \quad \boldsymbol{w}(x) = \mathrm{col}\,(u_{(1)}(x), \cdots, u_{(n)}(x)),$$

and

$$a = (a_{ij}) \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b = \mathrm{diag}\,(0, \cdots, 0, 1/p_n),$$

$$c = (c_{ij}) \quad \text{where} \quad c_{ij} = \begin{cases} p_{i-1} & \text{if } j = i \\ q_{i-1} & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$d = (d_{ij}) \quad \text{where} \quad d_{ij} = \begin{cases} -1 & \text{if } j = i - 1 \\ q_{n-1}/p_n & \text{if } j = i = n \\ 0 & \text{otherwise.} \end{cases}$$

Equation (1.12) allows an analogous recursive formulation

$$v_{(n)} = P_n v^{(n)}$$

$$v_{(n-k)} = -v'_{(n-k+1)} + Q_{n-k} v^{(n-k+1)} + P_{n-k} v^{(n-k)}, \qquad k = 1, 2, \cdots, n-1.$$

$$v'_{(1)} = Q_0 v' + P_0 v$$

which leads to the vector system

$$\text{(2.2)} \qquad \begin{aligned} \boldsymbol{v}' &= A(x)\,\boldsymbol{v} + B(x)\,\boldsymbol{z} \\ \boldsymbol{z}' &= C(x)\,\boldsymbol{v} + D(x)\,\boldsymbol{z} \end{aligned}$$

where

$$\boldsymbol{v}(x) = \operatorname{col}(v(x), v'(x), \cdots, v^{(n-1)}(x)) \quad , \quad \boldsymbol{z}(x) = \operatorname{col}(v_{(1)}(x), \cdots, v_{(n)}(x))$$

and the coefficient matrices are obtained from $a(x)$, $b(x)$, $c(x)$, $d(x)$ by replacing $p_i(x)$ by $P_i(x)$ and $q_i(x)$ by $Q_i(x)$.

The smallest $\beta > \alpha$ such that

$$u(\alpha) = u'(\alpha) = \cdots = u^{(n-1)}(\alpha) = 0 = u_{(1)}(\beta) = u_{(2)}(\beta) = \cdots = u_{(n)}(\beta)$$

is denoted by $\tilde{\mu}_1(\alpha)$ and called the *first focal point* of α with respect to (1.11). $\tilde{\eta}_1(\alpha)$ is defined analogously with respect to (1.12).

Considering the matrix systems

$$\text{(2.3)} \qquad \begin{aligned} U' &= a(x)\,U + b(x)\,W \\ W' &= c(x)\,U + d(x)\,W \end{aligned}$$

and

$$\text{(2.4)} \qquad \begin{aligned} V' &= A(x)\,V + B(x)\,Z \\ Z' &= C(x)\,V + D(x)\,Z \end{aligned}$$

where U, V, W, and Z are $n \times n$ matrices whose columns are solutions of (2.1) and (2.2), respectively, it is possible to relate the existence of focal points to singularities of specific solutions of (2.3) and (2.4). In particular, let $u_1(x), \cdots, u_n(x)$ be linearly independent solutions of (1.11) (or the more general recursive system) which satisfy $u_k^{(j)}(\alpha) = 0$ for $j = 0, \cdots, n-1$ and $u_{k(j)}(\alpha) = \delta_{jk}$ for $j, k = 1, \cdots, n$.

Then the matrices $U(x)$, $W(x)$, whose k–th columns are the vector solutions of (2.1) generated by $u_k(x)$, satisfy (2.3) and the initial conditions $U(\alpha) = 0$, $W(\alpha) = I$. Since *every* solution of (1.11) with an n–th order zero at $x = \alpha$ is a linear combination of $u_1(x), \cdots, u_n(x)$, it follows that $\tilde{\mu}_1(\alpha)$ is the smallest $\beta > \alpha$ such that the system

$$\sum_{k=1}^{n} c_k\, u_{k(j)}(\beta) = 0\,; \qquad\qquad j = 1, \cdots, n$$

has a nontrivial solution $c_1, \cdots, c_n$. That is, $\tilde{\mu}_1(\alpha)$ is the smallest $\beta > \alpha$ such that $W(\beta)$ is singular. The focal point $\tilde{\eta}_1(\alpha)$ has an analogous characterization in terms of a solution $V(x)$, $Z(x)$ of (2.4) satisfying $V(\alpha) = 0$, $Z(\alpha) = I$.

Consider now the transformation

$$M(x) = U(x)\, W^{-1}(x) \quad ; \quad N(x) = V(x)\, Z^{-1}(x) \tag{2.5}$$

where U, W and V, Z are the matrix solutions constructed above. The matrices $M(x)$ and $N(x)$ satisfy the matrix Riccati equations

$$M' = b(x) - Mc(x)M + a(x)M - Md(x) \tag{2.6}$$

and

$$N' = B(x) - NC(x)N + A(x)N - ND(x)\,, \tag{2.7}$$

respectively, with initial conditions $M(\alpha) = N(\alpha) = 0$. The solutions of (2.6) and (2.7) can be continued up to $\tilde{\mu}_1(\alpha)$ and $\tilde{\eta}_1(\alpha)$, respectively, and it is this latter characterization of focal points in terms of the interval of existence of solutions of (2.6) and (2.7) which underlies our principal result. We summarize this discussion as follows.

2.1. LEMMA. *The focal points $\tilde{\mu}_1(\alpha)$ and $\tilde{\eta}_1(\alpha)$ are the upper limits of the interval existence of the initial value problems* (2.6) *and* (2.7), *respectively.*

3. PRINCIPAL RESULTS

In this section we establish comparison theorems for focal points of (1.11) and (1.12).

3.1. THEOREM. *Let lu and Lv be defined by* (1.11) *and* (1.12) *respectively. If $l \geq L$ in the sense that*

$$\begin{aligned} &\text{(i)} \quad p_n(x) \geq P_n(x) > 0 && \\ &\text{(ii)} \quad 0 \geq p_k(x) \geq P_k(x) && k = 0, 1, \cdots, n-1 \\ &\text{(iii)} \quad 0 \geq q_k(x) \geq Q_k(x) && k = 0, 1, \cdots, n-1 \end{aligned} \tag{3.1}$$

for $\alpha \leq x \leq \tilde{\mu}_1(\alpha)$, then $\tilde{\mu}_1(\alpha) \geq \tilde{\eta}_1(\alpha)$.

Proof. This theorem follows from the theory of monotone Riccati operators developed by W. T. Reid [9]. For a square matrix $G = (G_{ij})$, the symbol $G \cdot \geq \cdot 0$ denotes $G_{ij} \geq 0$ for $i, j = 1, \cdots, n$, and $G \cdot \geq \cdot F$ if $G - F \cdot \geq \cdot 0$. Strict inequality is similarly defined. If

$$
(3.2) \quad
\begin{array}{ll}
\text{(i)} & B(x) \cdot \geq \cdot b(x) \cdot \geq \cdot 0 \quad ; \quad b(x) \not\equiv 0 \\
\text{(ii)} & -C(x) \cdot \geq \cdot - c(x) \cdot \geq \cdot 0 \\
\text{(iii)} & A(x) \cdot \geq \cdot a(x) \cdot \geq \cdot 0 \\
\text{(iv)} & -D(x) \cdot \geq \cdot - d(x) \cdot \geq \cdot 0 ,
\end{array}
$$

then the solutions of (2.5) and (2.6) with $M(\alpha) = N(\alpha) = 0$ satisfy $N(x) \cdot \geq \cdot M(x) \cdot \geq \cdot 0$ on the common interval of existence of these solutions.

Furthermore the hypotheses of the theorem assure that the conditions (3.2) are satisfied. From the Perron–Frobenius theory of positive matrices, it follows that the largest (necessarily positive) eigenvalue of $N(x)$ is at least as large as the largest (necessarily positive) eigenvalue of $M(x)$ and that the interval of existence of $M(x)$ is therefore at least as large as that of $N(x)$. By Lemma 2.1 it follows that $\tilde{\mu}_1(\alpha) \geq \tilde{\eta}_1(\alpha)$.

The question naturally arises whether the inequalities required in Theorem 3.1 can in any way be weakened. One basis for trying to weaken these hypotheses is the observation that the inequality $B(x) \cdot \geq \cdot b(x)$ in (3.2) can be replaced by the weaker condition

$$
(3.3) \qquad \int_{\alpha}^{x} B(\xi)\, d\xi \cdot \geq \cdot \int_{\alpha}^{x} b(\xi)\, d\xi > 0
$$

for $x > \alpha$. To see that this is sufficient to assure $N(x) \cdot \geq \cdot M(x)$ one need only apply the Picard iterations to (2.5) and (2.6) with initial choice $M_0(x) \equiv 0$, $N_0(x) \equiv 0$. Then one can replace the first inequality in (3.1) (i) by the weaker condition

$$
(3.4) \qquad \int_{\alpha}^{x} \frac{1}{P_n(\xi)}\, d\xi \geq \int_{\alpha}^{x} \frac{1}{p_n(\xi)}\, d\xi
$$

for $\alpha \leq x_1 \leq \tilde{\mu}_1(\alpha)$.

One might also hope to compare conjugate points on the basis of inequalities such as (3.2). Defining $M(x)$ and $N(x)$ by

$$
M(x) = W(x)\, U^{-1}(x) \quad ; \quad N(x) = Z(x)\, V^{-1}(x)
$$

instead of (2.5) would yield

$$
M' = c(x) - Mb(x)\, M + d(x)\, M - Ma(x) ,
$$
$$
N' = C(x) - NB(x)\, N + D(x)\, N - NA(x).
$$

Unfortunately the signs preceding the last two terms in both equations destroy the monotonicity of the associated Riccati operator.

In certain instances it is, however, possible to make a transformation eliminating these terms while retaining the desired monotonicity properties. In particular, if $P(x)$ is a nonsingular solution of $P' = AP$, then the substitution

$$U = PY \quad ; \quad W = P^{*-1}Z$$

transforms (2.3) into

$$\text{(3.5)} \qquad \begin{aligned} Y' &= P^{-1}\, b(x)\, P^{*-1} Z \\ Z' &= P^*\, c(x)\, PY - P^*(a^*(x) + d(x))\, P^{*-1} Z\,. \end{aligned}$$

In the special case of the equation

$$(p_2(x)\,y'')'' - (p_1(x)\,y')' + q_0(x)\,y' = 0$$

we can choose

$$P(x) = \begin{pmatrix} 1 & -x \\ 0 & -1 \end{pmatrix}$$

to get

$$P^{-1}\,bP^{*-1} = \begin{pmatrix} x^2/p_2 & x/p_2 \\ x/p_2 & 1/p_2 \end{pmatrix}$$

$$P^*cP = \begin{pmatrix} 0 & -q_0 \\ 0 & p_1 + xq_0 \end{pmatrix}$$

$$a^* + d = 0\,.$$

By requiring that $x \geq 0$, $q_0(x) \geq 0$, and $p_1(x) + xq_0(x) \leq 0$ one can apply the above monotonicity theory to the comparison of conjugate points of equations of the form (3.5).

4. A COUNTEREXAMPLE

The question remains whether the operator ordering in the hypotheses of Theorem 3.1 also determines an ordering of conjugate points. That this is not the case follows easily from the inequalities (1.6) applied to the second order equations (1.4) and (1.5). In particular, it is possible to have

$$\begin{aligned} p_1(x) &= P_1(x) > 0 \\ 0 &\geq p_0(x) \geq P_1(x) \\ 0 &= q_0(x) \geq Q_0(x) \end{aligned}$$

so that Theorem 3.1 implies $\tilde{\mu}_1(\alpha) \geq \tilde{\eta}_1(\alpha)$, while at the same time

$$\begin{aligned} P_1(x) &= p_1(x) > 0 \\ P_0(x) &> p_0(x) + \frac{1}{2}\frac{d}{dx}\,Q_0(x) \end{aligned}$$

so that the criterion (1.6) implies that $\mu_1(\alpha) < \eta_1(\alpha)$ with $\mu_1(\alpha)$ and $\eta_1(\alpha)$ denoting the first conjugate points of α with respect to (1.4) and (1.5), respectively.

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