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On spreads of curves

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Topologia. — *On spreads of curves.* Nota (*) di CONSTANTIN IVAN, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si stabiliscono risultati sulle famiglie continue di curve, mostrando fra l'altro l'esistenza in esse di al più un insieme numerabile di curve luoghi di punti di molteplicità $2k$, con $k \geq 1$. I casi $k = 1$ e $k = 2$ sono già stati rispettivamente considerati da Grünbaum e da Zamfirescu.

This Note concerns spreads of curves as introduced by Grünbaum in [1] (see also [2]). A spread is a family $\mathfrak{L}$ of Jordan arcs (further called curves), satisfying the following conditions:

i) each curve $L \in \mathfrak{L}$ (except its end-points) lies in the bounded component D of the complementary of a closed Jordan curve C and its end-points belong to C;

ii) each point $p \in C$ is the end-point of exactly one curve $L(p)$;

iii) if $L_1, L_2 \in \mathfrak{L}$ are two different curves, then $L_1 \cap L_2$ is a single point;

iv) the curve $L(p)$ depends continuously on $p \in C$.

Following [3], a maximal connected subset of $\mathfrak{L}$ the elements of which are concurrent curves is called a pencil.

Let $M_n(\mathfrak{L})$ be the set of all points of D which belong to at least n curves in $\mathfrak{L}$. Put $T_n(\mathfrak{L}) = M_n(\mathfrak{L}) - M_{n+1}(\mathfrak{L})$. The elements of $T_3(\mathfrak{L})$ are called triple points.

We have already a collection of results about "exceptions" that a spread may admit. Thus, Grünbaum [1] proved that on all curves of a spread, with at most one exception, there are triple points. Zamfirescu [3] completed this result by proving that on all curves with at most three exceptions there are non-degenerate arcs consisting of triple points (if $M_{\chi_0} = \varnothing$).

Consider now a spread $\mathfrak{L}$ without pencils. We ask:

"On how many curves $L \in \mathfrak{L}$, $\operatorname{int}(L \cap M_2(\mathfrak{L})) \subset T_j(\mathfrak{L})$, $(j \geq 2)$?"

The cited result of Grünbaum implies the answer "on at most one curve" for $j = 2$. Zamfirescu [4] proved that if $M_{\chi_0} = \varnothing$ then

$$\operatorname{int}(L \cap M_2(\mathfrak{L})) = L \cap T_4(\mathfrak{L})$$

on at most one curve L. From his Lemma 5 in [4] it follows that

$$\operatorname{int}(L \cap M_2(\mathfrak{L})) \subset T_4(\mathfrak{L})$$

implies

$$\operatorname{int}(L \cap M_2(\mathfrak{L})) = L \cap T_4(\mathfrak{L}) .$$

(*) Pervenuta all'Accademia il 24 luglio 1973.

Also, his proof works if $M_{\chi_0} = \varnothing$ is replaced by the condition that $\mathfrak{L}$ has no pencils; thus his result implies the same answer to our question for $j = 4$ as for $j = 2$.

In this Note we investigate the problem in the general even case.

THEOREM. *If $\mathfrak{L}$ is a spread without pencils, then*

$$\operatorname{int}(L \cap M_2(\mathfrak{L})) \subset T_{2k}(\mathfrak{L}), \qquad (k \geq 1)$$

for at most countably many curves $L \in \mathfrak{L}$.

Let $f : (a, b) \to R$ be a continuous bounded function such that

i) for every $\lambda \in \operatorname{int} f((a, b))$, $\operatorname{card} f^{-1}(\lambda) = 2k - 1$, where k is an arbitrary positive fixed integer;

ii) for every $\mu \in \operatorname{fr} f((a, b))$, $f^{-1}(\mu)$ contains no interval.

LEMMA 1. *Every point $x \in (a, b)$ is a strict relative extreme for the restrictions of the function f to the intervals $(a, x]$ and $[x, b)$.*

This Lemma coincides with Lemma 2 from [4] (stated for $k = 2$) and admits the same proof.

The point $x \in (a, b)$ is said to be of type $(+,+)$ (respectively $(+,-)$, $(-,-)$, $(-,+)$) if it is simultaneously a strict relative maximum (respectively maximum, minimum, minimum) for $f_{/(a,x]}$ and a strict relative maximum (respectively minimum, minimum, maximum) for $f_{/[x,b)}$ [4].

Let $x_i \in (a, b)$, $i = 1, 2, \cdots, 2k - 1$, be $2k - 1$ points such that

$$a < x_1 < x_2 < \cdots < x_{2k-1} < b$$

and

$$f(x_1) = f(x_2) = \cdots = f(x_{2k-1}) \in \operatorname{int} f((a, b)).$$

Evidently, if x_i is a relative maximum (respectively minimum) for $f_{/[x_i,b)}$, then x_{i+1}, $i = 1, 2, \cdots, 2k - 2$ must necessarily be a relative maximum (respectively minimum) for $f_{/(a,x_{i+1}]}$ too. Therefore there are 2^{2k} possible sequences of these types for the points x_i, $i = 1, 2, \cdots, 2k - 1$.

It is also easy to show that, from these 2^{2k} possible sequences only those containing as many points of type $(+,+)$ as of type $(-,-)$ may actually appear and that the numbers of points of type $(+,-)$ and $(-,+)$ differ by one.

LEMMA 2. *If $\mu \in \operatorname{fr} f((a, b))$, then* $\operatorname{card} f^{-1}(\mu) = k - 1$.

This is a generalization of Lemma 5 from [4] (stated for $k = 2$) and can be proved in the same way.

LEMMA 3. *Let $F \subset f((a, b))$ be the set of all μ's such that at least one of the points of $f^{-1}(\mu)$ is of type $(+,+)$. Then F is at most countable.*

Proof. Let $\mu \in F \cap \operatorname{int} f((a, b))$ and $\{x_1, \cdots, x_{2k-1}\} = f^{-1}(\mu)$ and suppose x_j is of type $(+,+)$. Let I be a subinterval of (a, b) between a point of type $(-,+)$ or $(+,+)$ and another point of type $(+,-)$ or $(+,+)$.

There are exactly $k-1$ such intervals $I_1, I_2, \cdots, I_{k-1}$. Also, either x_1 is of type $(+,-)$ or $(+,+)$, or x_{2k-1} is of type $(-,+)$ or $(+,+)$. In the first case denote $I_0 = (a, x_1)$, in the second $I_0 = (x_{2k-1}, b)$. Now it is easily seen that each point of $f^{-1}(\lambda)$, where

$$\lambda \in (\mu, \min_i \max_{x \in I_i} f(x)),$$

is neither of type $(+,+)$ nor of type $(-,-)$. The fact that F is countable is now obvious.

Let $\mathfrak{L}$ be a spread without pencils and let $L(p) \in \mathfrak{L}$ be such that

$$\operatorname{int}(L(p) \cap M_2(\mathfrak{L})) \subset L(p) \cap T_{2k}(\mathfrak{L}).$$

Consider two homeomorphisms

$$\varphi : [a, b] \to A$$
$$\psi : [c, d] \to L(p)$$

providing parametric representations of the curve $L(p)$ with the end-points p and $-p$ and of one of the two arcs of C determined by these end-points, A.

Then the application

$$f : (a, b) \to (c, d)$$

defined by

$$f(x) = \psi^{-1}(L(\varphi(x)) \cap L(p))$$

is continuous. Also,

$$f((a, b)) = \psi^{-1}(L(p) \cap M_2(\mathfrak{L}))$$

and

$$\psi^{-1}(L(p) \cap T_{2k}(\mathfrak{L})) = \{\lambda \in f(a, b) : \operatorname{card} f^{-1}(\lambda) = 2k-1\}.$$

Since

$$\operatorname{int}(L(p) \cap M_2(\mathfrak{L})) \subset L(p) \cap T_{2k}(\mathfrak{L}),$$
$$\operatorname{int} f((a, b)) \subset \{\lambda \in f(a, b) : \operatorname{card} f^{-1}(\lambda) = 2k-1\}$$

and since $\mathfrak{L}$ has no pencils, for each $\mu \in f((a, b))$, $f^{-1}(\mu)$ includes no interval. Hence f is a function of the type investigated above.

LEMMA 4. *Let z be an arbitrary point belonging to* $\operatorname{int}(L(p) \cap M_2(\mathfrak{L}))$, *such that the subset $f^{-1}(\psi^{-1}(z))$ of (a, b) contains only points of the types $(+,-)$ and $(-,+)$. Then on every curve $L \in \mathfrak{L}$ passing through z and different from $L(p)$ there are points through which pass at least $2k+1$ curves.*

Proof. Let $f^{-1}(\psi^{-1}(z)) = \{x_1, x_2, \cdots, x_{2k-1}\}$. Suppose for instance that $\varphi(a) = \psi(c) = p$ and $\varphi(b) = \psi(d) = -p$,

$$x_0 = a < x_1 < x_2 < \cdots < x_{2k-1} < b = x_{2k}$$

and that the point x_1 is of type $(+,-)$, the other case being analogous.

Let $L = L(\varphi(x_1))$, the proof for the other curves $L(\varphi(x_i))$ $(i = 1, 2, \cdots, 2k-1)$ being similar. Then there exist $k+1$ points $y_0 \in (x_0, x_1)$

and $y_i \in (x_{2i-1}, x_{2i})$ $(i = 1, 2, \cdots, k)$, such that $L(\varphi(y_0)) \cap L(p)$ lies between p and z on $L(p)$, and $L(\varphi(y_i)) \cap L(p)$ $(i = 1, 2, \cdots, k)$ lie between z and $-p$ on $L(p)$. Therefore, evidently, $L(\varphi(y_i)) \cap L$ $(i = 0, 1, 2, \cdots, k)$ are between z and $-\varphi(x_1)$ on L.

Let z' be a point of the curve L simultaneously placed between z and $L \cap L(\varphi(y_i))$ $(i = 0, 1, 2, \cdots, k)$. Then, in accordance with Lemma 1 of [1], there exist $2k$ points $p_1^1 \in (x_0, y_1)$, $p_1^2 \in (y_2, x_2)$, $p_i^1 \in (x_{2i-1}, y_i)$ and $p_i^2 \in (y_i, x_{2i})$ $(i = 2, 3, \cdots, k)$ such that $z' \in L(\varphi(p_i^j))$ $(i = 1, 2, \cdots, k$ and $j = 1, 2)$.

Then $z' \in L \cap M_{2k+1}(\mathfrak{L})$.

The following lemma is a generalization of Lemma 7 in [4].

LEMMA 5. *For every curve* $L \in \mathfrak{L}$ *different from* $L(p)$ *intersecting* $\operatorname{fr}(L(p) \cap M_2(\mathfrak{L}))$, $\operatorname{int}(L \cap M_2(\mathfrak{L})) - M_{k+1}(\mathfrak{L}) \neq \varnothing$.

Proof. Let z be the common point of $L = L(x_0)$ and $L(p)$. Because $\varphi^{-1}(z) \in \operatorname{fr} f((a, b))$, by Lemma 2, $\operatorname{card} f^{-1}(\varphi^{-1}(z)) \leq k - 1$. Then $z \in M_2(\mathfrak{L}) - M_{k+1}(\mathfrak{L})$.

Suppose that all interior points of $L(p) \cap M_2(\mathfrak{L})$ lie on $L(p)$ between z and $-p$. Then there exist two points $x_1 \in (a, x_0)$ and $x_2 \in (x_0, b)$ (different from the end-points of all other $k-2$ curves passing through z), such that the point z lies between $L \cap L(\varphi(x_1))$ and $L \cap L(\varphi(x_2))$ on the curve L. It follows that $z \in \operatorname{int}(L \cap M_2(\mathfrak{L}))$ and then $z \in \operatorname{int}(L \cap M_2(\mathfrak{L})) - M_{k+1}(\mathfrak{L})$.

Now the proof of the theorem reduces to an obvious combination of Lemmas 3, 4 and 5.

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