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Congruence conditions for Riemannian N-manifolds with groups of motions

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Fisica matematica. — *Congruence conditions for Riemannian N-manifolds with groups of motions* (*). Nota (**) di CARLO MOROSI (***), presentata dal Socio B. FINZI.

RIASSUNTO. — Si determinano, nel caso di piccole deformazioni e per varietà riemanniane N–dimensionali con gruppi di moto, lo spostamento indotto da una deformazione congruente, nonché le condizioni necessarie e sufficienti di congruenza per la deformazione stessa.

1. INTRODUCTION

The congruence conditions are the necessary and sufficient conditions for a second-rank symmetric tensor to be the deformation δa_{ik} induced on the metric tensor a_{ik} of a manifold V_N by a displacement field s_i; thus in the case of small deformations they are the integrability conditions of the tensor equation

$$\xi_{ik} \equiv \delta a_{ik} = s_{i/k} + s_{k/i} \qquad (i , k = 1 , 2 , \cdots , N) . \tag{1.1}$$

These conditions have been obtained for Riemannian manifolds V_N with no group of motions [1], for which the homogeneous tensor equation

$$\delta a_{ik} = v_{i/k} + v_{k/i} = 0 \qquad (i , k = 1 , 2 , \cdots , N) \tag{1.2}$$

has the trivial solution only.

As for Riemannian manifolds V_N with groups of motions, a method to obtain necessary, but generally not sufficient, congruence conditions has been shown [2], and the analysis of the congruence in the particular case of a rotation surface has been completed [3]; the method used in [3] is generalized to Riemannian N-manifolds in this paper. Therefore N-manifolds with groups of motions are considered, that is with rigid infinitesimal displacements (solutions of Eq. (1.2)) parallel to r $(1 \leq r \leq N)$ linearly independent congruences; as a particular case $(r = N)$, Euclidean manifolds are obtained.

The congruence conditions are given by making zero the congruence functions, that are obtained in tensor form and are linear functions of the strain tensor ξ_{ik} and its tensor derivatives; furthermore they are linked by $(N - r)$ linear and differential identities; as these identities cannot be reduced to finite identities among the congruence functions, they do not allow to reduce

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the number of essential congruence conditions. The method used to obtain the congruence conditions suggests how to obtain a particular displacement field induced on V_N by any congruent strain tensor: hence by adding the Killing vector we obtain the most general displacement field solution of Eq. (1.1); differently from the case of no group of motions [1], the displacement field is here obtained by means of integration and differential operators, and can be actually calculated by using a particular coordinate system set up on the manifold.

2. N-MANIFOLDS WITH GROUPS OF MOTIONS

The congruence conditions for a surface applicable on a rotation surface can be obtained in two equivalent ways [3], either by requiring that the differential invariants H and L (1) be the same functions of the Gaussian curvature K both for the deformed manifold and for the undeformed one [5, Ch. 3], or by projecting the vector field s_i onto two congruences of the manifold given by the curvature gradient $K_{/i}$ and the (unique) Killing vector v_i:

$$s_i \equiv \underset{k}{s}\, K_{/i} + \underset{v}{s}\, v_i \,. \tag{2.1}$$

In this case the congruence conditions are the very existence conditions of the scalar invariants $\underset{k}{s}$ and $\underset{v}{s}$.

The second procedure will be generalized to the case of N-manifolds in this paper; to this end the explicit determination of the congruence conditions will be preceded by a short analysis of the representation of tensor objects (in particular vectors) defined on the manifold: that is, a suitable representation of the form (2.1) is looked for. Therefore we consider a Riemannian N-manifold with a group of rigid motions, that is with rigid motions alongside $r\,(1 \leq r \leq N)$ linearly independent congruences (Killing congruences); thus the general solution v_i of Eq. (1.2) can be represented by a linear combination of r linearly independent solutions of Eq. (1.2), in the form (2)

$$v_i \equiv \underset{n}{a}\, \underset{n}{v_i} \,. \tag{2.2}$$

These vectors characterize a submanifold $V_r \subset V_N$ that we call "Killing manifold"; however we remark that not all the vectors (2.2), even if belonging

(1) The invariants H and L are differential invariants (of the first and the second order respectively) of the Gaussian curvature K of the surface; they are defined as follows:

$$H \equiv K^{/i}\, K_{/i} \,,\; L = K_{/i}^{\,i} \,.$$

(2) From now on, Latin suffixes ($n = 1, 2, \cdots, r$) and Greek suffixes ($\alpha = r + 1, \cdots, N$) are to be summed if they are repeated; the case $r = 0$ is not analyzed in this paper, being fully treated in [1].

to V_r, are Killing vectors, solutions of Eq. (1.2). Furthermore we can define, upon V_N, $(N-r)$ functionally independent invariants $\underset{\alpha}{A}$ $(\alpha = r+1, \cdots, N)$ functions of the metric tensor of the manifold and its ordinary derivatives: therefore we have $(N-r)$ linearly independent vectors $\underset{\alpha}{A}_{/i}$ and a submanifold $V_{N-r} \subset V_N$ spanned by these vectors.

Now the set of N vectors $\{\underset{\alpha}{A}_{/i} ; \underset{n}{v}_i\}$ can be chosen as a vector basis of V_N; in fact the r vectors $\underset{n}{v}_i$ and the $(N-r)$ vectors $\underset{\alpha}{A}_{/i}$ are linearly independent by construction, and, moreover, for any vector $u_i \in V_{N-r}$ and $K_i \in V_r$

$$u_i K^i = 0 \tag{2.3}$$

that is the two manifolds V_r and V_{N-r} are orthogonal. This property follows from the fact that the Killing congruences characterize the directions of the rigid motions, for which $\delta a_{ik} = 0$, hence we have

$$\delta \underset{\alpha}{A} \equiv \underset{\alpha}{A}(a + \delta a) - \underset{\alpha}{A}(a) = \underset{\alpha}{A}_{/i} \underset{n}{v}^i = 0 \qquad (\forall \alpha , n) \tag{2.4}$$

from what Eq. (2.3) follows, u_i and K_i being linear combinations of $\underset{\alpha}{A}_{/i}$ and $\underset{n}{v}_i$ respectively. Furthermore it follows from (2.4) that for any displacement vector $K_i \in V_r$, generally non-rigid (that is with $\delta a_{ik} \neq 0$),

$$\delta \underset{\alpha}{A} \equiv \underset{\alpha}{A}(a + \delta a) - \underset{\alpha}{A}(a) = \underset{\alpha}{A}_{/i} K^i = 0 \tag{2.4'}$$

that is the functional variation of any invariant constructed with the metric tensor vanishes on the Killing manifold [6, Ch. 12]. Therefore in correspondence to any choice of a vector basis $\{\underset{\alpha}{A}_{/i} ; \underset{n}{v}_i\}$ the manifold is decomposed as the sum of two orthogonal submanifolds

$$V_N = V_r \oplus V_{N-r} . \tag{2.5}$$

In spite of the fact that the invariants $\underset{\alpha}{A}$ cannot be chosen univocally, as well as the Killing vectors $\underset{n}{v}_i$ (the parameters of the group of motions being generally more than r), nevertheless the Killing manifold V_r and its orthogonal complement V_{N-r} are univocally determined: for that the decomposition (2.5) is invariant. Therefore, even if a complete system of " intrinsic coordinates " cannot be set up (differently from the case of no rigid motions [1]), for any tensor object we can consider its components alongside the particular congruences given by the chosen vector basis: we shall name these particular components, even if improperly, " *intrinsic components* ". In particular we consider vectors and second-rank tensors: a vector field of V_N (e.g. the eventual solution of Eq. (1.1) we are looking for) can be given the following form:

$$s_i \equiv u_i + K_i \equiv \underset{\alpha}{s} \underset{\alpha}{A}_{/i} + \underset{n}{s} \underset{n}{v}_i \tag{2.6}$$

where

$$u_i \equiv \underset{\alpha}{s} \underset{\alpha}{A}_{/i} \quad ; \quad K_i \equiv \underset{n}{s} \underset{n}{v}_i \tag{2.7}$$

are respectively the vector components of s_i lying on the manifolds V_{N-r} and V_r. The decomposition of a second-rank symmetric tensor is the following

$$\sigma_{ik} \equiv \underset{\alpha\beta}{\sigma} \underset{\alpha}{A}_{/i} \underset{\beta}{A}_{/k} + \underset{\alpha m}{\sigma} (\underset{\alpha}{A}_{/i} \underset{m}{v}_k + \underset{\alpha}{A}_{/k} \underset{m}{v}_i) + \underset{mn}{\sigma} \underset{m}{v}_i \underset{n}{v}_k . \tag{2.8}$$

Of course the manifold V_N can be characterized by a different vector basis; in particular we can consider a basis $\{\underset{\alpha}{A}'_{/i} ; \underset{n}{v}'_i\}$ constructed by means of invariants $\underset{\alpha}{A}'$ and Killing vectors $\underset{n}{v}'_i$ defined as follows

$$\underset{\alpha}{A}' \equiv \varphi_\alpha (\underset{\beta}{A}) \quad ; \quad \underset{n}{v}'_i \equiv \underset{nm}{c} \underset{m}{v}_i \tag{2.9}$$

where the functions φ_α do not depend on particular properties of V_N: that is, if $\underset{\alpha}{\bar{A}}'$ and $\underset{\alpha}{\bar{A}}$ are the invariants of another manifold V'_N (e.g. obtained from V_N by a generic strain), it is always $\underset{\alpha}{\bar{A}}' = \varphi_\alpha (\underset{\beta}{\bar{A}})$ (3).

By such a choice, the intrinsic components $\underset{\alpha}{s}'$ and $\underset{n}{s}'$ are linear combinations of the first ones: furthermore they are scalar objects, that is invariant under coordinate mappings; instead the decomposition (2.6) of the vector $\vec{s}$ as the sum of the vectors $\vec{u} \in V_{N-r}$ and $\vec{K} \in V_r$ is invariant, that is

$$u'_i = u_i \quad ; \quad K'_i = K_i \tag{2.10}$$

referring to their components in any reference frame.

As it will be shown in the next section, a vector field induced on the manifold by the strain tensor can be obtained by means of the decomposition (2.6); in fact, the intrinsic components $\underset{\alpha}{s}$ can be obtained by means of any strain ξ_{ik}, even if not congruent, and for any choice of the invariants $\underset{\alpha}{A}$ (and they are unique only if the strain is congruent), whereas the determination of the intrinsic components $\underset{n}{s}$ is linked in an essential way to the congruence of the strain tensor.

(3) Differently from the case outlined above, a functional relation like (2.9) may be valid for particular manifolds only: the typical case is that of a rotation surface (see footnote (1)) where $H = f(K)$, while for a generic surface H and K are functionally independent.

3. DETERMINATION OF THE DISPLACEMENT FIELD AND OF THE CONGRUENCE CONDITIONS

In this section a vector field is obtained (in correspondence to any infinitesimal and *congruent* strain ξ_{ik}) that is induced by ξ_{ik} upon V_N: furthermore, by requiring that this field be solution of Eq. (1.1), we obtain the necessary and sufficient congruence conditions for the strain tensor.

Firstly we consider (for a strain ξ_{ik} generally not congruent), the system

$$\delta \underset{\alpha}{A} \equiv \underset{\alpha}{A}(a+\xi) - \underset{\alpha}{A}(a) = \underset{\alpha}{A}_{/i}\, \varphi^i : \tag{3.1}$$

its infinite many solutions φ^i are

$$\varphi^i \equiv \underset{\alpha\beta}{B}\, \delta \underset{\beta}{A}\, \underset{\alpha}{A}^{/i} + w^i \tag{3.2}$$

where $\underset{\alpha\beta}{B}$ is the inverse matrix of the matrix $\underset{\alpha\beta}{A} \equiv \underset{\alpha}{A}_{/i} \underset{\beta}{A}^{/i}$ and w^i is a generic vector of V_r. In particular if $w^i = 0$, that is if the particular solution $\varphi_0^i \in V_{N-r}$ is chosen, one has

$$\varphi_0^i \equiv \underset{\alpha\beta}{B}\, \delta \underset{\beta}{A}\, \underset{\alpha}{A}^{/i}. \tag{3.3}$$

The following properties of φ^i and φ_0^i are of some interest: firstly, φ^i and φ_0^i can always be obtained, by Eq. (3.1), for any strain ξ_{ik}, even if not congruent, and φ_0^i does not depend upon a change of vector basis such like (2.9), as follows from (3.3). Instead, if $(N-r)$ invariants $\underset{\alpha}{\bar{A}} = \bar{\varphi}_\alpha(\underset{\beta}{A})$ are considered on the undeformed manifold V_N, a vector $\bar{\varphi}_0^i \neq \varphi_0^i$ is generally obtained; but if the strain ξ_{ik} is congruent, and *only in this case*, we have

$$\delta \underset{\alpha}{\bar{A}} = \frac{\partial \bar{\varphi}_\alpha}{\partial \underset{\beta}{A}}\, \delta \underset{\beta}{A} \tag{3.4}$$

hence

$$\varphi_0^i = \bar{\varphi}_0^i ; \tag{3.5}$$

in this case φ_0^i is univocally determined. Therefore (3.4) (as well as (3.5)) are necessary, but generally not sufficient, congruence conditions. At last, if the strain ξ_{ik} is congruent (hence Eq. (1.1) has a solution), φ_0^i $(\varphi_0^i = \bar{\varphi}_0^i)$ is the vector component on V_{N-r} of the solutions of Eq. (1.1); in fact by (2.3) and (2.6)

$$\delta \underset{\alpha}{A}(\xi) = \underset{\alpha}{A}_{/i}\, s^i = \underset{\alpha}{A}_{/i}\, u^i \tag{3.6}$$

hence, by (3.1) and (2.7)

$$\varphi_0^i = u^i : \tag{3.7}$$

in what follows the vector defined by (3.3) will be indicated by u^i. Thus, on account of (2.7) and (3.3) the "intrinsic components" $\underset{\alpha}{s}$ of the *eventual* solution of Eq. (1.1) can be given the form

$$\underset{\alpha}{s} \equiv \underset{\alpha\beta}{B}\,\delta\underset{\beta}{A} \tag{3.8}$$

(that is they are linear functions of the variations $\delta\underset{\beta}{A}$), whether or not the strain ξ_{ik} is congruent.

Now we come back to our first assumption: the strain ξ_{ik} is congruent, hence Eq. (1.1) has a solution with the $(N - r)$ intrinsic components $\underset{\alpha}{s}$ given by (3.8); our aim is to obtain the remaining r components $\underset{n}{s}$. To this end Eq. (1.1) can be given, by (2.6), the following form

$$\xi_{ik} = u_{i/k} + u_{k/i} + (\underset{n}{s}\,\underset{n}{v_i})_{/k} + (\underset{n}{s}\,\underset{n}{v_k})_{/i} \;; \tag{3.9}$$

if the tensor

$$\sigma_{ik} \equiv \xi_{ik} - (u_{i/k} + u_{k/i}) \tag{3.10}$$

is defined (by (3.3) and (3.7), σ_{ik} is a linear function of the strain and its tensor derivatives) and Eq. (1.2) is used, Eq. (3.9) becomes

$$\sigma_{ik} = \underset{n}{s}_{/i}\,\underset{n}{v_k} + \underset{n}{s}_{/k}\,\underset{n}{v_i} \tag{3.11}$$

that is we have a linear system for the gradients $\underset{n}{s}_{/i}$; on account of the assumed congruence of the strain ξ_{ik} the system (3.11) has a solution and the gradients $\underset{n}{s}_{/i}$ can be given the form

$$\underset{n}{s}_{/i} = \underset{n\alpha}{\Lambda}\,\underset{\alpha}{A}_{/i} + \underset{nl}{M}\,\underset{l}{v_i} \,. \tag{3.12}$$

Thus the strain ξ_{ik} satisfies a set of particular conditions: the "irrotationality" conditions of (3.12)

$$\underset{n}{C}_{ik} \equiv (\underset{n\alpha}{\Lambda}\,\underset{\alpha}{A}_{/i} + \underset{nl}{M}\,\underset{l}{v_i})_{/k} - (\underset{n\alpha}{\Lambda}\,\underset{\alpha}{A}_{/k} + \underset{nl}{M}\,\underset{l}{v_k})_{/i} = 0 \,. \tag{3.13}$$

The relations (3.13) have been obtained as necessary congruence conditions: are they also sufficient, that is are the N invariants $\underset{\alpha}{s}$ and $\underset{n}{s}$ the very intrinsic components of a solution of Eq. (1.1)?

Firstly we point out that the conditions (3.13) are actually *sufficient* in order that a vector field be induced upon V_N by ξ_{ik}; in fact by (3.3), (3.7) and (3.10) the tensor σ_{ik} is obtained for any infinitesimal strain ξ_{ik}: then if $\underset{nl}{N}$ and $\underset{nl}{P}$ are the symmetric and skew-symmetric parts of the matrix $\underset{nl}{M}$

$$\underset{nl}{M} \equiv \underset{nl}{N} + \underset{nl}{P} \tag{3.14}$$

the following expressions of $\underset{n\alpha}{\Lambda}$ and $\underset{nl}{\mathrm{N}}$ are obtained by means of (3.12) and (3.11)

$$\underset{n\alpha}{\Lambda} = \sigma_{ik} \underset{\gamma}{\mathrm{A}}^{/i} \underset{m}{v}^{k} \underset{\gamma\alpha}{\mathrm{B}} \underset{mn}{\mathrm{W}} \tag{3.15}$$

$$\underset{nl}{\mathrm{N}} = \sigma_{ik} \underset{m}{v}^{i} \underset{s}{v}^{k} \underset{mn}{\mathrm{W}} \underset{sl}{\mathrm{W}} \tag{3.16}$$

where $\underset{mn}{\mathrm{W}}$ is the inverse matrix of the matrix $\underset{mn}{\mathrm{V}} \equiv \underset{m}{v}^{i} \underset{n}{v}_{i}$: of course the skew-symmetric elements $\underset{nl}{\mathrm{P}}$ cannot be obtained, for σ_{ik} is symmetric *by construction*.

Thus Eq. (3.12) can be solved in correspondence to any choice of $\underset{nl}{\mathrm{P}}$ for which the r tensors $\underset{n}{\mathrm{C}}_{ik}$ vanish, and r scalars $\underset{n}{s}$ are obtained: on account of (3.8), by which $\underset{\alpha}{s}$ are given without requiring further conditions upon ξ_{ik}, N invariants $(\underset{\alpha}{s}, \underset{n}{s})$ are determined as linear functions of ξ_{ik}; therefore a vector field induced by the strain upon V_{N} is obtained with intrinsic components $\underset{\alpha}{s}$ and $\underset{n}{s}$.

In order that Eq. (1.1) be solved, we have to verify whether or not the vector field $s_i(\xi_{mn})$ just obtained is actually a solution of Eq. (1.1), that is whether or not the tensor

$$\eta_{ik}(\xi_{mn}) \equiv \xi_{ik} - (s_{i/k} + s_{k/i}) \tag{3.17}$$

or (with reference to the decomposition (2.8)) its intrinsic components $\underset{\alpha\beta}{\eta}, \underset{mn}{\eta}, \underset{\alpha n}{\eta}$ vanish identically; in fact one can verify, by means of (3.11), (3.12) and (3.13), that $\underset{\alpha n}{\eta} = \underset{mn}{\eta} = 0$, whereas $\underset{\alpha\beta}{\eta} \neq 0$.

Therefore the only set of conditions (3.13) is not sufficient in order that (3.17) be valid: the strain ξ_{ik} must satisfy also to the following set of conditions

$$\underset{\alpha\beta}{\eta} = \underset{\alpha\beta}{\sigma} \equiv \sigma_{ik} \underset{\alpha}{\mathrm{A}}^{/i} \underset{\beta}{\mathrm{A}}^{/k} = 0 \tag{3.18}$$

(that can be formally obtained directly by (3.11) and (3.12)).

Thus the necessary and sufficient congruence conditions are given by the two sets (3.13) and (3.18): by the first one a field $s_i(\xi_{mn})$ can be obtained induced by the strain, and if the strain satisfies also to the second one this field is a particular solution of Eq. (1.1). At last if the general solution v_i of the homogeneous equation (1.2) is added, the general solution of the complete equation is obtained.

We remark that an actual integration of the system (3.12) is required for the determination of the components $\underset{n}{s}$; for that a particular coordinate system must be chosen and operators of integration introduced, in contrast with the case of the components $\underset{\alpha}{s}$ that can be expressed, by means of (3.8), in an invariant form as functions of the strain and its tensor derivatives.

Finally we stress that, despite the fact that the largely arbitrary elements $\underset{nl}{P}$ are used, the solutions of Eq. (1.1) do not depend upon them; in fact, even if these elements could not be chosen so as to reduce the number of essential congruence conditions (as will be indicated in the following section), or they were not univocally determined by the conditions (3.13), one can easily show that the vector fields obtained in correspondence to different determinations of $\underset{ln}{P}$ differ by a rigid displacement, but, as already stressed, this indetermination is just the characteristic property of the manifolds we are analyzing on account of the existence of non-trivial solutions of Eq. (1.2).

4. Congruence conditions

Before discussing some qualitative features of the congruence conditions just obtained, we give the conditions (3.13) an invariant tensor form, as well as the conditions (3.18); in fact the (3.13) are equivalent to

$$(4.1)\qquad \begin{cases} \underset{s\alpha\beta}{C} \equiv \frac{1}{2}(\underset{s}{C_{ik}} - \underset{s}{C_{ki}})\, \underset{\alpha}{A}^{/i}\, \underset{\beta}{A}^{/k} = 0 & (\alpha \neq \beta) \\ \underset{smn}{C} \equiv \frac{1}{2}(\underset{s}{C_{ik}} - \underset{s}{C_{ki}})\, \underset{m}{v}^{i}\, \underset{n}{v}^{k} = 0 & (m \neq n) \\ \underset{s\alpha n}{C} \equiv \frac{1}{2}(\underset{s}{C_{ik}} - \underset{s}{C_{ki}})\, \underset{\alpha}{A}^{/i}\, \underset{n}{v}^{k} = 0\,. & \end{cases}$$

Therefore they express that the non-trivial intrinsic components of the r tensors $\underset{s}{C_{ik}}$ are zero; in this way the congruence conditions are obtained by making zero $\underset{\alpha\beta}{\sigma}, \underset{s\alpha\beta}{C}, \underset{smn}{C}, \underset{s\alpha n}{C}$, that are linear functions of the strain tensor ξ_{ik} and its tensor derivatives. Moreover if another vector basis $\{\underset{\alpha}{A'_{/i}}\,;\,\underset{n}{v'_i}\}$ is chosen, the new congruence functions are linear combinations of the first ones, and they are equivalent to them, being zero if and only if the first ones too are zero. Another feature of the congruence functions is given by the fact that they are differentially linked; in fact for any strain (independent of whether the conditions (3.13) and (3.18) are verified) the tensor σ_{ik} satisfies *by construction* to the following $(N - r)$ identities

$$(4.2)\qquad \delta\underset{\alpha}{A}(\sigma_{ik}) \equiv \delta\underset{\alpha}{A}(\xi_{ik}) - \delta\underset{\alpha}{A}(u_{i/k} + u_{k/i}) = 0$$

that are linear differential identities for σ_{ik}: the highest order of derivation depends upon the particular choice of the invariants $\underset{\alpha}{A}$.

Even if the above identities can be written as identities among the congruence functions, the number of essential congruence conditions cannot in general be reduced, as well as in the case of a rotation surface [3], by means of the identities (4.2), for no homogeneous and finite identity can be obtained from them among the congruence functions. At last we point out that the $rN(N-1)/2$ functions (4.1) just obtained contain $r(r-1)/2$ arbitrary invariants $\underset{nl}{P}$; on account of this arbitrariness not less than $r(r-1)/2$ congruence functions can be made zero for any strain: as the congruence functions defined by (3.13) are $(N-r)(N-r+1)/2$, the maximum number $\mathcal{C}(N;r)$ of congruence conditions is given by

$$\mathcal{C}(N;r)=\frac{N(N+1)}{2}-\frac{rN(3-N)}{2}\,. \tag{4.3}$$

As the conditions (4.1), that contain $\underset{nl}{P}$, are differential of the first order and generally integrable in $\underset{nl}{P}$, other arbitrary functions have to be introduced when $\underset{nl}{P}$ are actually obtained from them, hence in some cases more congruence functions than $r(r-1)/2$ can be made zero for any strain: thus the essential congruence conditions may be less than $\mathcal{C}(N;r)$ for particular manifolds. Without any detailed analysis of particular cases, we simply remark that $\mathcal{C}(N;r)=3-r$ for the surface; therefore, as well known [3, 7], there is only one congruence condition $(r=2)$ for a surface with zero or constant Gaussian curvature, and there are two essential congruence conditions $(r=1)$ for a surface applicable on a rotation surface.

5. CONCLUSION

The displacement field has been obtained, for Riemannian N-manifolds with groups of motions, induced by a congruent strain (in the case of small deformations), as well as the necessary and sufficient congruence conditions for the strain. As suggested by [8], [4] and applied in [1], the knowledge of the congruence conditions allows to obtain the solution of the " equilibrium " equations

$$p^{ki}{}_{/k}=0 \quad ; \quad p^{ki}{}_{/k}=f^i \tag{5.1}$$

when the equilibrium problem is the adjoint problem of the congruence [2]: this problem will be analyzed in a forthcoming paper.

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