Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Shri Krishna Singh, Sangappa Mallappa Sarangi

On Nevanlinna Modified Deficiency

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **55** (1973), n.5, p. 341–347.

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<http://www.bdim.eu/item?id=RLINA_1973_8_55_5_341_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1973.

Funzioni meromorfe. — *On Nevanlinna Modified Deficiency.* Nota di SHRI KRISHNA SINGH e SANGAPPA MALLAPA SARANGI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori provano che se $f(z)$ è una funzione meromorfa nel piano complesso, se la somma delle sue deficienze generalizzate è uguale a 2 e se per qualche valore di r la deficienza risulta uguale ad 1, allora l'ordine di $f(r)$ è un intero positivo.

Let $f(z)$ be a meromorphic function of order ρ $(0 \leq \rho \leq \infty)$. Let α be a non-negative number such that $\alpha < \rho$ if $\rho \neq 0$ and $\alpha = 0$ if $\rho = 0$. Let $T(r,f)$, $m(r,a)$, $N(r,a)$, $\bar{N}(r,a)$, $\delta(a,f)$, $\Theta(a,f)$, $S(r,f)$, etc. have the usual meaning in Nevanlinna theory. Following Toda [1] we define for any $r_0 > 0$

$$T_\alpha(r,f) = \int_{r_0}^{r} \frac{T(t,f)}{t^{1+\alpha}}\, dt$$

$$N_\alpha(r,a,f) = N_\alpha(r,a) = \int_{r_0}^{r} \frac{N(t,a)}{t^{1+\alpha}}\, dt$$

$$\bar{N}_\alpha(r,a,f) = \bar{N}_\alpha(r,a) = \int_{r_0}^{r} \frac{\bar{N}(t,a)}{t^{1+\alpha}}\, dt$$

$$\delta_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{N_\alpha(r,a)}{T_\alpha(r,f)}$$

$$\Theta_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)} .$$

We call $\delta_\alpha(a,f)$ modified α-deficiency in the sense of Nevanlinna, and $\Theta_\alpha(a,f)$ α-deficiency for distinct a-points of $f(z)$.

Then it is known that for all $a \in \bar{\mathbf{C}}$ $\delta(a,f) \leq \delta_\alpha(a,f)$ and the set $N_\alpha = \{a\,;\ \delta_\alpha(a) > 0\}$ is countable and $\sum_{a\in\bar{\mathbf{C}}} \delta_\alpha(a,f) \leq 2$ see [1].

We prove the following.

THEOREM. *If $f(z)$ is a transcendental meromorphic function of finite order such that $\sum_{a\in\bar{\mathbf{C}}} \delta_\alpha(a,f) = 2$ and if for some $b \in \bar{\mathbf{C}}$, $\delta_\alpha(b,f) = 1$, then ρ is a positive integer.*

(*) Nella seduta del 26 novembre 1973.

COROLLARY. *If for some positive integer* k, $\sum_{a \in \overline{\mathbf{C}}} \delta_\alpha(a, f^{(k)}) = 2$, *then* ρ *is a positive integer.*

For the proof of the above theorem we shall need the following.

LEMMA. *Let* $f(z)$ *be a meromorphic function of non-integral order* ρ $(0 < \rho < \infty)$ *let* $\alpha < \rho$ *and let*

$$K_\alpha(\rho) = \limsup_{r \to \infty} \frac{N_\alpha(r, 1/f) + N_\alpha(r, f)}{T_\alpha(r, f)} . \tag{1}$$

Then

$$K_\alpha(\rho) \geq 1 - \rho \qquad \text{if} \quad 0 < \rho < 1 \tag{2}$$

and

$$K_\alpha(\rho) \geq \frac{(q + 1 - \rho)(\rho - q)}{\rho c_1(q)} \qquad \text{if} \quad \rho > 1,\ q = [\rho] \tag{3}$$

where

$$c_1(q) = 2(q+1)(2 + \log q) \qquad \text{if} \quad q \geq 1$$

$$c_1(q) = 1 \qquad \text{if} \quad q = 0 .$$

The above lemma is a slightly improved version of theorem 3 of Toda [1]. We shall give an alternative proof of this lemma covering the case $\rho = 0$ namely we shall prove that $K_0(0) \geq 1$.

Proof. First of all we shall prove that for $\alpha < \rho$, $0 < \rho < \infty$,

$$\limsup_{r \to \infty} \frac{\log T_\alpha(r, f)}{\log r} = \rho - \alpha . \tag{4}$$

(4) is again due to Toda, however our proof is different.

The fact that $\limsup_{r \to \infty} \dfrac{\log T_\alpha(r, f)}{\log r} \geq \rho - \alpha$ is a simple consequence of $T_\alpha(kr, f)\, r^\alpha \geq A T(r, f)$ for $k > 1$. For the inequality in the other direction, we define $\rho(r)$ as a proximate order relative to $T(r, f)$. Then $\rho(r) \to \rho$ as $r \to \infty$, $r\rho'(r) \log r \to 0$ as $r \to \infty$,

$$T(r, f) \leq r^{\rho(r)} \quad \text{for } r \geq r_0 \quad \text{and} \quad T(r, f)' = r^{\rho(r)} \quad \text{for a sequence of } r \to \infty$$

since $0 < \rho < \infty$, such a $\rho(r)$ does exist, see [2, p. 35].

Hence

$$T_\alpha(r, f) = \int_{r_0}^{r} \frac{T(t, f)}{t^{1+\alpha}} \, dt \leq \int_{r_0}^{r} t^{\rho(t) - \alpha - 1} \, dt \sim \frac{r^{\rho(r) - \alpha}}{\rho - \alpha}$$

so

$$\limsup_{r \to \infty} \frac{\log T_\alpha(r, f)}{\log r} \leq \rho - \alpha \quad (\text{since } \rho(r) \to \rho).$$

This proves (4).

Now it is known that if $f(z)$ is a meromorphic function of order ρ and genus q, then it can be expressed as

$$f(z) = z^m e^{Q(z)} (\Pi/\mu) E\left(\frac{z}{a_\mu}, q\right) \Big/ (\Pi/\nu) E\left(\frac{z}{a_\nu}, q\right)$$
$$= z^m e^{Q(z)} \frac{P_1}{P_2} \quad \text{say}$$

with the same q in P_1 and P_2, since ρ is non-integer, $q = [\rho]$ is the genus of $f(z)$.

Hence following the usual method it is easy to prove that

$$\log M(r, P_1) \leq A(q) \left\{ qr^q \int_0^r \frac{n(t, 0)}{t^{q+1}} dt + (q+1)_r^{q+1} \int_r^\infty \frac{n(t, 0)}{t^{q+2}} dt \right\}$$

where $A(q) = 1$ if $q = 0$

$$A(q) = 2(2 + \log q) \qquad \text{if} \quad q \geq 1.$$

Similar inequality holds for $\log M(r, P_2)$ replacing $n(t, 0)$ by $n(t, \infty)$. Combining these inequalities, noting that $T(r, P_1) \leq \log M(r, P_1)$ and integrating by parts, we get

$$\text{(5)} \qquad T(r, f) \leq A(q) \left\{ qr^q \int_0^r \frac{N(t)}{t^{q+1}} dt + (q+1)_r^{q+1} \int_r^\infty \frac{N(t)}{t^{q+2}} dt \right\} + o(r^q)$$

where

$$A(q) = 2(q+1)(2 + \log q) \qquad \text{if} \quad q \geq 1$$
$$A(q) = 1 \qquad \text{if} \quad q = 0,$$

and $N(t) = N(t, 0) + N(t, \infty)$.

Hence from (5) dividing by $r^{1+\alpha}$ and integrating by parts from r_0 to r, we get for $q \geq 1$

$$\text{(6)} \qquad T_\alpha(r, f) \leq A(q) \left\{ qr^{q-\alpha} \int_{r_0}^r \frac{N_\alpha(t)}{t^{q+1-\alpha}} dt + \right.$$
$$\left. + (q+1) r^{q+1-\alpha} \int_r^\infty \frac{N_\alpha(t)}{t^{q+2-\alpha}} dt \right\} + o(r^{q-\alpha}).$$

Let $\rho_1(0)$ and $\rho_1(\infty)$ be the exponent of convergence formed with the zeros and poles of $f(z)$. Then since ρ is non-integer, $\rho = \operatorname{Max}\{\rho_1(0), \rho_1(\infty)\}$. Now $N(r) = N(r, 0) + N(r, \infty) \leq 2T(r, f) + o(1)$ hence

$$\limsup_{r \to \infty} \frac{\log N(r)}{\log r} \leq \rho.$$

On the other hand,

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \geq \operatorname{Max}\{\rho_1(0), \rho_1(\infty)\} = \rho$$

hence

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \rho .$$

Now following exactly as in the proof of (4) we deduce that

$$\limsup_{r\to\infty} \frac{\log N_\alpha(r)}{\log r} = \rho - \alpha . \tag{7}$$

Hence there exists a proximate order $\rho(r)$ relative to $N_\alpha(r)$ such that

$$\begin{aligned} &\rho(r) \to \rho - \alpha && \text{as} \quad r \to \infty \\ &r\rho'(r)\log r \to 0 && \text{as} \quad r \to \infty \\ &N_\alpha(r) \leq r^{\rho(r)} && \text{for} \quad r \geq r_0 . \\ &N_\alpha(r) = r^{\rho(r)} && \text{for a sequence of} \quad r \to \infty . \end{aligned}$$

Hence

$$T_\alpha(r,f) \leq A(q)\left\{ q r^{q-\alpha} \int_{r_0}^{r} t^{\rho(t)-q-1-\alpha}\, dt + \right.$$

$$\left. + (q+1) r^{q+1-\alpha} \int_{r}^{\infty} t^{\rho(t)-q-2+\alpha}\, dt \right\} + o(r^{q-\alpha})$$

$$\sim A(q)\left\{ q r^{q-\alpha} \frac{r^{\rho(r)-q+\alpha}}{\rho-\alpha-q+\alpha} + \frac{(q+1) r^{q+1-\alpha}\, r^{\rho(r)-q-1+\alpha}}{q+1-\rho} \right\} + o(r^{q-a})$$

$$= A(q)\left\{ q \frac{r^{\rho(r)}}{\rho-q} + (q+1)\frac{r^{\rho(r)}}{q+1-\rho} \right\} + o(r^{\rho(r)})$$

(since $q-\alpha < \rho-\alpha$ and $\rho(r) \to \rho-\alpha$).

Now, since $N_\alpha(r) = r^{\rho(r)}$ for a sequence of $r \to \infty$, we get

$$\liminf_{r\to\infty} \frac{T_\alpha(r,f)}{N_\alpha(r)} \leq A(q)\left\{ \frac{q}{\rho-q} + \frac{q+1}{q+1-\rho} \right\} = \frac{\rho A(q)}{(\rho-q)(q+1-\rho)} .$$

If $q = 0$, then $0 < \rho < 1$. Hence in this case from the inequality

$$T(r,f) \leq \int_0^r \frac{n(t)}{t}\, dt + r\int_r^\infty \frac{n(t)}{t^2}\, dt + o(\log r)$$

we deduce,

$$T(r,f) \leq r\int_r^\infty \frac{N(t)}{t^2}\, dt + o(\log r) .$$

Hence, dividing by $r^{1+\alpha}$ and integrating by parts from r_0 to r, we get

$$\mathrm{T}_\alpha(r,f) \leq r^{1-\alpha} \int_r^\infty \frac{\mathrm{N}_\alpha(t)}{t^{2-\alpha}}\, \mathrm{d}t + o(1)$$

$$\mathrm{T}_\alpha(r,f) \leq r^{1-\alpha} \int_r^\infty t^{\rho(t)-2+\alpha}\, \mathrm{d}t + o(1)$$

$$\sim r^{1-\alpha} \frac{r^{\rho(r)-1+\alpha}}{1-\alpha-(\rho-\alpha)}$$

$$= \frac{r^{\rho(r)}}{1-\rho} .$$

Hence,

$$\limsup_{r\to\infty} \frac{\mathrm{N}_\alpha(r)}{\mathrm{T}_\alpha(r,f)} \geq 1-\rho .$$

Finally, if $\rho = 0$, then

$$\limsup_{r\to\infty} \frac{\log \mathrm{N}_\alpha(r)}{\log r} = 0$$

hence,

$$\limsup_{r\to\infty} \frac{\log\{r^m \mathrm{N}_\alpha(r)\}}{\log r} = m , \qquad \text{for} \quad m > 0$$

so there exists a proximate order $\rho(r)$ relative to $r^m \mathrm{N}_\alpha(r)$ such that

$$\rho(r) \to m \qquad \text{as} \quad r \to \infty$$

$$r\rho'(r) \log r \to 0 \qquad \text{as} \quad r \to \infty$$

(8) $$r^m \mathrm{N}_\alpha(r) \leq r^{\rho(r)} \qquad \text{for} \quad r \geq r_0$$

(9) $$r^m \mathrm{N}_\alpha(r) = r^{\rho(r)} \qquad \text{for a sequence of } r \to \infty .$$

Assuming that $f(0) = 1$, which we can do without loss of generality, we have

$$\mathrm{T}(r,f) \leq r \int_r^\infty \frac{\mathrm{N}(t)}{t^2}\, \mathrm{d}t$$

hence

$$\mathrm{T}_\alpha(r,f) \leq r \int_r^\infty \frac{\mathrm{N}_\alpha(t)}{t^2}\, \mathrm{d}t$$

$$\leq r \int_r^\infty t^{\rho(t)-(m+2)}\, \mathrm{d}t \quad \text{from (8)}$$

$$\sim r \frac{r^{\rho(r)-m-1}}{m+1-m} = r^{\rho(r)-m}$$

$$= \mathrm{N}_\alpha(r) \text{ for a sequence of } r \to \infty \text{ from (9).}$$

Hence

$$\liminf_{r \to \infty} \frac{T_\alpha(r,f)}{N_\alpha(r)} \leq 1 .$$

Thus $K_0(0) \geq 1$. This completes the proof.

Note. Since $T\left(r, \frac{Af+B}{cf+D}\right) = T(r, f) + o(1)$, $N_\alpha(r, 1/f)$ and $N_\alpha(r, f)$ in (1) can be replaced by $N_\alpha(r, a)$ and $N_\alpha(r, b)$ $a, b \in \overline{\mathbf{C}}$, $a \neq b$.

The above argument also provides an alternative proof of the fact that for meromorphic function of order zero,

$$\limsup_{r \to \infty} \frac{N(r,a) + N(r,b)}{T(r,f)} \geq 1 \qquad \text{for all} \quad a, b \in \overline{\mathbf{C}}, (a \neq b).$$

Proof of the theorem. Without any loss of generality we can assume that $b = \infty$, since otherwise consider $F(z) = \frac{1}{f(z) - b}$. Let $\{a_i\}_{i=1}^{\infty}$ be distinct elements of $\mathbf{C}$ which include all those $a \in \mathbf{C}$ for which $\delta_\alpha(a, f) > 0$. Thus

$$(10) \qquad \delta_\alpha(\infty, f) = 1$$

$$(11) \qquad \sum_{i=1}^{\infty} \delta_\alpha(a_i, f) = 1 .$$

Now from [1, thm 1] we have

$$\sum_{a \neq \infty} \delta_\alpha(a) \leq \liminf_{r \to \infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} \leq \limsup_{r \to \infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} \leq 2 - \Theta_\alpha(\infty)$$

so from (10) and $\delta_\alpha(\infty, f) \leq \Theta_\alpha(\infty, f)$ we get

$$\lim_{r \to \infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} = 1 .$$

Again from [1; thm 2]

$$\sum_{a \neq \infty} \delta_\alpha(a) \leq \delta_\alpha(0, f')(2 - \Theta_\alpha(\infty))$$

hence from (10) and (11)

$$1 \leq \delta_\alpha(0, f')(2 - \delta_\alpha(\infty)) \leq \delta_\alpha(0, f')$$

so $\delta_\alpha(0, f') = 1$.

Hence

$$N_\alpha(r, 1/f') = o(T_\alpha(r, f')).$$

Also

$$N_\alpha(r, \infty, f') \leq 2 N_\alpha(r, \infty, f) = o(T_\alpha(r, f)) = o(T_\alpha(r, f')).$$

Hence

$$K_\alpha(f') = 0 .$$

So from the lemma the order of $f'(z)$ is a positive integer. But since the order of $f'(z)$ is the same as the order of $f(z)$, it follows that order of $f(z)$ is a positive integer.

Proof of Corollary. Let $f^{(k)}(z) = F(z)$

$$N(r, \infty, F) \geq 2\,\overline{N}(r, \infty, F)$$

so

$$N_\alpha(r, \infty, F) \geq 2\,\overline{N}_\alpha(r, \infty, F).$$

Hence

$$2\,\Theta_\alpha(\infty, F) - \delta_\alpha(\infty, F) \geq 1. \tag{12}$$

But since $\delta_\alpha(a, F) \leq \Theta_\alpha(a, F)$ for all $a \in \overline{\mathbf{C}}$ and since $\sum_{a \in \overline{\mathbf{C}}} \Theta_\alpha(a, F) \leq 2$, the hypothesis $\sum_{a \in \overline{\mathbf{C}}} \delta_\alpha(a, F) = 2$ immediately gives $\delta_\alpha(a, F) = \Theta_\alpha(a, F)$ for all $a \in \overline{\mathbf{C}}$.

In particular, $\Theta_\alpha(\infty, F) = \delta_\alpha(\infty, F)$.

Hence from (12) we get $\delta_\alpha(\infty, F) = \Theta_\alpha(\infty, F) = 1$. So

$$\sum_{a \neq \infty} \delta_\alpha(a, F) = 1 \qquad \text{and} \qquad \delta_\alpha(\infty, F) = 1.$$

Hence by the theorem, the order of F, hence that of f is a positive integer.

Note. Since $\delta(a, f) \leq \delta_\alpha(a, f)$ for all $a \in \overline{\mathbf{C}}$, from the above theorem and the corollary we deduce that if $\delta(b, f) = 1$, $\sum_{i=1}^{\infty} \delta(a_i, f) = 1$ then ρ must be a positive integer and if $\sum_{i=1}^{\infty} \delta(a_i, f^{(k)}(z)) = 2\ (k \geq 1)$ then ρ must be a positive integer. Let us note that we cannot go the other way, since there do exist meromorphic functions for which $\delta(a, f) < \delta_\alpha(a, f)$ see [1].

REFERENCES

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