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On a classical theorem concerning algebraic systems of hypersurfaces in a projective space or in a projective variety

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Geometria. — *On a classical theorem concerning algebraic systems of hypersurfaces in a projective space or in a projective variety* (*). Nota di UMBERTO BARTOCCI (**), presentata (***) dal Socio B. SEGRE.

RIASSUNTO. — Viene stabilito in ogni caratteristica l'analogo del cosiddetto *Teorema dell'Indice* per sistemi algebrici di ipersuperficie di uno spazio proiettivo o di una varietà proiettiva (cfr. B. Segre [9] per il caso della caratteristica zero) e ne vengono tratte alcune applicazioni.

I. - INTRODUCTION

We shall deal with a projective r–dimensional ($r \geq 1$) space $\mathbf{P}^r$ defined over any algebraically closed field K of characteristic $p \geq 0$, and choose once for all a *universal domain* Ω over K (we shall always use the language of André Weil's « Foundations of Algebraic Geometry » — quoted as FAG — for *varieties and cycles, generic points, general theory of intersection multiplicity*). $\mathbf{P}^N$, $N = \binom{n+r}{r} - 1$, will designate the projective N–dimensional space representing all hypersurfaces of $\mathbf{P}^r$ of degree n (≥ 1), and $L_x \subset \mathbf{P}^N$, for each point $x \in \mathbf{P}^r$, will be the *Veronese hyperplane* corresponding to x in $\mathbf{P}^N$, i.e. the set of all hypersurfaces of degree n containing x [the hyperplanes L_x form an algebraic variety isomorphic to $\mathbf{P}^r$ in the dual space $\overset{*}{\mathbf{P}}{}^N$ of $\mathbf{P}^N$, the so-called *Veronese variety* of indices (r, n)]. Finally, $\mathfrak{D}$ will indicate the K–closed subset of $\mathbf{P}^N$ representing all hypersurfaces of degree n with *multiple components*.

DEFINITION 1. - Let $V \subseteq \mathbf{P}^N$ be any δ–dimensional subvariety of $\mathbf{P}^N$ of degree m defined over K. We shall call the *algebraic system* of hypersurfaces associated with V, and denote it by $\Sigma = \Sigma(V)$, the set of all hypersurfaces of $\mathbf{P}^r$ corresponding to points of V. δ and m will be, respectively, the *dimension* and the *degree* of Σ.

Remark - For the sake of simplicity we shall always suppose throughout the first three sections that Σ *has no fixed components*. Furthermore, we recall that such a system is often called an *irreducible* algebraic system (see [9] (1) or [14]).

From now on we suppose δ to be greater than zero.

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(1) The numbers between [ ] refer to the Bibliography at the end of the paper.

DEFINITION 2. – The *index* ν of Σ is the number of distinct points in the o–dimensional cycle $(L_{\xi_1} \cap \cdots \cap L_{\xi_\delta}) \cdot_{\mathbf{P}^N} V$—write $\nu = \#[(L_{\xi_1} \cap \cdots \cap L_{\xi_\delta}) \cdot_{\mathbf{P}^N} V]$—where $\xi_1 \times \cdots \times \xi_\delta$ is a generic point of $\underbrace{\mathbf{P}^r \times \cdots \times \mathbf{P}^r}_{\delta \text{ times}}$ over K.

From this definition it follows that $\nu \leq m$, and it may be shown that there exist algebraic systems such that $\nu < m$:

Example 1. – Write the *generic conic* of the projective plane $\mathbf{P}^2$ as

$$a_{00}\, x_0^2 + a_{11}\, x_1^2 + a_{22}\, x_2^2 + a_{01}\, x_0\, x_1 + a_{02}\, x_0\, x_2 + a_{12}\, x_1\, x_2 = 0,$$

and consider the 1–dimensional algebraic system Σ associated with the variety V of equations $a_{22} = a_{02} = a_{12} = 0$, $a_{01}^2 - 4\, a_{00}\, a_{01} = 0$ (suppose $p \neq 2$). The degree of Σ is 2 but its index is 1.

The so-called Index Theorem in the case $K = \mathbf{C}$—complex number field—asserts that the index ν is equal to the degree m provided that V is not contained in $\mathfrak{Y}$, say Σ *has no variable multiple components* (see for instance [9]; the first proofs are in [2], [4], [5], 10]). This theorem is generally proved by means of differential methods. We want to give here a complete algebraic proof of this proposition, which does not give any additional information if $p = 0$ in view of the so-called *Lefschetz principle* (FAG, p. 306), but which contributes also in the case of positive characteristic furnishing *a necessary and sufficient condition* for the equality $m = \nu$.

Needless to say, the following is a rather elementary application of the general methods of the theory of algebraic correspondences as developed by André Weil in FAG and in the fundamental paper [14] and by Oscar Zariski in many papers but specifically in the first chapter of [16].

II. – THE INDEX THEOREM FOR ALGEBRAIC SYSTEMS OF HYPERSURFACES IN A PROJECTIVE SPACE

Before giving the more general statement of the Index Theorem in any characteristic we need some further notation.

As before let $\Sigma = \Sigma(V)$ be any algebraic system of hypersurfaces of $\mathbf{P}^r$, and let δ be equal to 1 (substantially only for typographical reasons, as we shall see later). Consider the product variety $\mathbf{P}^r \times \mathbf{P}^N$ and its subvariety W defined by

$$W = \{x \times \eta \in \mathbf{P}^r \times \mathbf{P}^N \,|\, x \in |f_\eta|\},$$

where f_η is the hypersurface of $\mathbf{P}^r$ corresponding to the point $\eta \in \mathbf{P}^N$ and $|f_\eta|$ its *support*.

W is a non–singular subvariety of $\mathbf{P}^r \times \mathbf{P}^N$ of dimension $r + N - 1$, and it is clear that, for each $\eta \in \mathbf{P}^N$,

$$pr_{\mathbf{P}^r}[(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times \mathbf{P}^N} W] = f_\eta,$$

where $pr_{\mathbf{P}^r}$ is the *projection map* $pr_{\mathbf{P}^r} : \mathbf{P}^r \times \mathbf{P}^N \rightarrow \mathbf{P}^r$.

Consider now the K–closed subset $Z_0 = W \cap (\mathbf{P}^r \times V)$ in $\mathbf{P}^r \times \mathbf{P}^N$. It follows immediately from the fact that $\mathbf{P}^r \times V$ is not contained in W (and that $\mathbf{P}^r \times \mathbf{P}^N$ is not singular) that Z_0 has only r–dimensional irreducible components (FAG, VI, Theorem 1, Corollary 1) and so W and $\mathbf{P}^r \times V$ *meet properly* in $\mathbf{P}^r \times \mathbf{P}^N$. Introduce the r–dimensional cycle

$$Z = W \cdot_{\mathbf{P}^r \times \mathbf{P}^N} (\mathbf{P}^r \times V) = \sum_{j=1}^{\rho} d_j Z_j ,$$

where $Z_1, \cdots, Z_\rho$ are the irreducible components of Z and of course $d_j = i(W \cdot (\mathbf{P}^r \times V), Z_j ; \mathbf{P}^r \times \mathbf{P}^N)$ (according to the notation of FAG, VI).

LEMMA 1. – *With the previous notation, let* η *be a simple point of* V *(possibly defined over an extension of* K). *Then*

$$f_\eta = \sum_{j=1}^{\rho} d_j \, pr_{\mathbf{P}^r} [(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times V} Z_j] .$$

Proof. $\mathbf{P}^r \times \eta$ and Z_j meet properly in $\mathbf{P}^r \times V$ provided that η is a simple point of V, and so we get

$$\sum_{j=1}^{\rho} d_j \, pr_{\mathbf{P}^r} [(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times V} Z_j] = pr_{\mathbf{P}^r} \left[(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times V} \sum_{j=1}^{\rho} d_j Z_j \right] =$$

$$= pr_{\mathbf{P}^r} [(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times V} (W \cdot_{\mathbf{P}^r \times \mathbf{P}^N} (\mathbf{P}^r \times V))] = pr_{\mathbf{P}^r} [(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times \mathbf{P}^N} W] = f_\eta$$

(using FAG, VI, Theorem 9).

From Lemma 1 it follows in particular that each map $\varphi_j : Z_j \to \mathbf{P}^r$ induced by the projection map $pr_{\mathbf{P}^r}$ is a *finite morphism*, because Σ has no fixed components. We shall indicate with φ_j^* the corresponding map between the function fields, $\varphi_j^* : K(\mathbf{P}^r) \to K(Z_j)$, and we shall say that φ_j is *separable* if the extension $K(Z_j) \supseteq \varphi_j^*(K(\mathbf{P}^r))$ is separable.

We can now formulate the Index Theorem as follows:

INDEX THEOREM (case $\delta = 1$). – *With the notation as before, the equality* $m = \nu$ *holds provided that*:

(i) Σ *has no variable multiple components*;

(ii) *each morphism* $\varphi_j : Z_j \to \mathbf{P}^r$ *is separable.*

Proof. Let us take a generic point ξ of $\mathbf{P}^r$ over K and consider the 0–dimensional cycle $L_\xi \cdot_{\mathbf{P}^N} V = \sum_{l=1}^{\nu} i(L_\xi \cdot V, \eta_l ; \mathbf{P}^N) \, \eta_l$. The points $\eta_1, \cdots, \eta_\nu$ are ν distinct generic points of V over K defined over $\overline{K(\xi)}$, the algebraic closure of $K(\xi)$ in Ω, and so each of these, say η_l, is such that the point $\xi \times \eta_l$ is a generic point over K of one—and then only one—of the varieties Z_j, say $Z_{j(l)}$.

We shall prove without any assumption about Σ [namely (i) and (ii)] that the identity

$$(1) \qquad i(L_\xi \cdot V, \eta_l ; \mathbf{P}^N) = d_{j(l)} \, p^{f_{j(l)}}$$

holds, where $p^{f_{j(l)}}$ is the inseparability degree of $K(Z_{j(l)})$ over $\varphi^*_{j(l)}(K(\mathbf{P}^r))$. Then the conclusion will follow easily, because the hypothesis (i) implies that each d_j is equal to 1 (from Lemma 1, with η as the generic point η_l of V over K), while the hypothesis (ii) implies that each p^{f_j} is equal to 1.

In order to establish (1), we begin by proving that

$$i(L_\xi \cdot V, \eta_l; \mathbf{P}^N) = d_{j(l)}\, i((\xi \times \mathbf{P}^N) \cdot Z_{j(l)}, \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N).$$

In fact we have

$$\begin{aligned} & d_{j(l)}\, i((\xi \times \mathbf{P}^N) \cdot Z_{j(l)}, \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i((\xi \times \mathbf{P}^N) \cdot d_{j(l)} Z_{j(l)}, \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i((\xi \times \mathbf{P}^N) \cdot Z, \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N), \end{aligned}$$

because $\xi \times \eta_l$ belongs only to $Z_{j(l)}$ among the varieties $Z_1, \cdots, Z_\rho$, and so we get

$$\begin{aligned} & i((\xi \times \mathbf{P}^N) \cdot Z, \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i((\xi \times \mathbf{P}^N) \cdot (W \cdot (\mathbf{P}^r \times V)), \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i(((\xi \times \mathbf{P}^N) \cdot W) \cdot (\mathbf{P}^r \times V), \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i((\xi \times L_\xi) \cdot (\mathbf{P}^r \times V), \xi \times \eta_l; \mathbf{P}^r \times \mathbf{P}^N) = \\ & = i(L_\xi \cdot V, \eta_l; \mathbf{P}^N) \end{aligned}$$

(using FAG, VI, Theorem 8: needless to say, the projection map $\xi \times L_\xi \to L_\xi$ is an isomorphism, and so in particular is *biregular* in η_l).

Finally we recall that we know that the 0-dimensional cycle $(\xi \times \mathbf{P}^N) \cdot_{\mathbf{P}^r \times \mathbf{P}^N} Z_{j(l)}$ contains a number of points equal to the separability degree of $K(Z_{j(l)})$ over $\varphi^*_{j(l)}(K(\mathbf{P}^r))$, write $\# [(\xi \times \mathbf{P}^N) \cdot_{\mathbf{P}^r \times \mathbf{P}^N} Z_{j(l)}] = [K(Z_{j(l)}) : \varphi^*_{j(l)}(K(\mathbf{P}^r))]_{\mathrm{sep}}$; moreover each of those points appears in this cycle with multiplicity equal to $p^{f_{j(l)}}$, from the assumption that ξ is a generic point of $\mathbf{P}^r$ over K (FAG, VI, Theorem 12), and $\xi \times \eta_l$ is one of these points, q.e.d.

We remark that the identity (1) implies

$$m = \sum_{j=1}^{\rho} d_j\, p^{f_j} [K(Z_j) : \varphi_j^*(K(\mathbf{P}^r))]_{\mathrm{sep}} = \sum_{j=1}^{\rho} d_j [K(Z_j) : \varphi_j^*(K(\mathbf{P}^r))], \tag{2}$$

$$\nu = \sum_{j=1}^{\rho} [K(Z_j) : \varphi_j^*(K(\mathbf{P}^r))]_{\mathrm{sep}}, \tag{3}$$

and so we get:

PROPOSITION 1. – *With the notation as before, the equality $m = \nu$ holds if and only if Z is a simple cycle and each morphism φ_j is separable.*

We now proceed to establish the Index Theorem for any δ. We recall that, when $p = 0$, a proof may be given by using a simple induction argument (see [9]). In fact, suppose $\delta > 1$ and that the theorem is true for the dimension $\delta - 1$. Then the theorem is true in particular for the generic hyperplane section $V' = V \cdot_{\mathbf{P}^N} L$ of V, defined over some extension K′ of K (recall that we know from the IInd Theorem of Bertini—see [9] or [17]—that V′ is an irreducible cycle), since V′ is of course not contained in $\mathfrak{V}$. Let $\xi_1 \times \cdots \times \xi_{\delta-1}$ be a generic point of $\underbrace{\mathbf{P}^r \times \cdots \times \mathbf{P}^r}_{\delta-1 \text{ times}}$ over K′; by the inductive hypothesis we know that $\nu' = \#[(L_{\xi_1} \cap \cdots \cap L_{\xi_{\delta-1}}) \cdot_{\mathbf{P}^N} V']$ is equal to the degree of V′, which is still m. Consider then the 1-dimensional cycle $V_0 = (L_{\xi_1} \cap \cdots \cap L_{\xi_{\delta-1}}) \cdot_{\mathbf{P}^N} V$. It is clear that V_0 is a simple cycle, because

$$\# [V_0 \cdot_{\mathbf{P}^N} L] = \# [(L_{\xi_1} \cap \cdots \cap L_{\xi_{\delta-1}}) \cdot_{\mathbf{P}^N} (V \cdot_{\mathbf{P}^N} L)] = m,$$

and that V_0 is not contained in $\mathfrak{V}$. So, again by induction, we get

$$\# [L_{\xi_\delta} \cdot_{\mathbf{P}^N} V_0] = m$$

and then

$$\# [(L_{\xi_1} \cap \cdots \cap L_{\xi_{\delta-1}} \cap L_{\xi_\delta}) \cdot_{\mathbf{P}^N} V] = m,$$

for any generic point ξ_δ of $\mathbf{P}^r$ over K′ such that $\xi_1 \times \cdots \times \xi_{\delta-1} \times \xi_\delta$ is a generic point of $\underbrace{\mathbf{P}^r \times \cdots \times \mathbf{P}^r \times \mathbf{P}^r}_{\delta \text{ times}}$ over K′, q.e.d.

Remark. – In the previous argument we have used a more general definition of algebraic system, by referring to simple cycles of $\mathbf{P}^N$; it is straightforward to extend all the previous assertions to this case.

When p is positive, the single hypothesis that Σ has no variable multiple components is not sufficient for the equality $m = \nu$, as seen above. Anyway we can state and prove the Index Theorem in all generality employing the same kind of argument given for the 1-dimensional case, with obvious slight modifications.

In fact we have to consider the product variety $\underbrace{\mathbf{P}^r \times \cdots \times \mathbf{P}^r}_{\delta \text{ times}} \times \mathbf{P}^N$ and then its subvariety W defined by

$$W = \{x_1 \times \cdots \times x_\delta \times \eta \in \mathbf{P}^r \times \cdots \times \mathbf{P}^r \times \mathbf{P}^N \,|\, x_i \in |f_\eta| \; \forall i\}.$$

W is a non-singular subvariety of $\mathbf{P}^r \times \cdots \times \mathbf{P}^r \times \mathbf{P}^N$ of dimension $r\delta + N - \delta$, and we can introduce as before the $r\delta$ – dimensional cycle

$$Z = W \cdot_{\mathbf{P}^r \times \cdots \times \mathbf{P}^r \times \mathbf{P}^N} (\mathbf{P}^r \times \cdots \times \mathbf{P}^r \times V) = \sum_{j=1}^{\rho} d_j Z_j .$$

For each Z_j we have a finite morphism $\varphi_j : Z_j \to \mathbf{P}^r \times \cdots \times \mathbf{P}^r$ induced by the projection map $pr_{\mathbf{P}^r \times \cdots \times \mathbf{P}^r} : \mathbf{P}^r \times \cdots \times \mathbf{P}^r \times \mathbf{P}^N \to \mathbf{P}^r \times \cdots \times \mathbf{P}^r$, and we can prove the:

INDEX THEOREM (General case). – *With the notation as before, the equality* $m = \nu$ *holds provided that*:

(i) Σ *has no variable multiple components.*

(ii) *each morphism* $\varphi_j : Z_j \to \mathbf{P}^r \times \cdots \times \mathbf{P}^r$ *is separable.*

Needless to say, one can extend the validity of formulae (2) and (3), and so the δ–dimensional formulation of Proposition 1 still holds.

Remark. – It is important to point out that the condition (ii) *does not* follow from the condition (i) and so the classical Index Theorem is no longer true for positive characteristic.

Example 2. – For any $\delta \geq 1$ consider on the projective line the following algebraic system depending on δ independent parameters $t_1, \cdots, t_\delta$

$$(x^2 - (t_1 + t_1^p)\, x + t_1^{p+1}) \cdots (x^2 - (t_\delta + t_\delta^p)\, x + t_\delta^{p+1}) = 0.$$

The identity $x^2 - (t + t^p)\, x + t^{p+1} = (t^p - x)(t - x)$ shows that $\nu = 2^\delta$, $m = (p + 1)^\delta > \nu$, in spite of the fact that the above system has no variable multiple components [in other words, consider e.g. the case $\delta = 1$: the variety V representing that system in a space $\mathbf{P}^2$ is a rational curve of degree $p + 1$ and its *dual curve* in $\overset{*}{\mathbf{P}}{}^2$ is exactly the Veronese variety of indices (1,2)].

III. – FURTHER REMARKS AND APPLICATIONS

Formulae (2) and (3) provide some arithmetical criteria for the equality $m = \nu$ when $\rho = 1$: for instance, the equality holds provided that Σ has no variable multiple components and p does not divide m, or else when m and n are coprime and again p does not divide m. But we want to show that this case $\rho = 1$ is really a special one, since:

PROPOSITION 2. – *The equality $m = \nu$ holds for any algebraic system Σ without variable multiple components and such that $\rho = 1$.*

COROLLARY 1. – *The equality $m = \nu$ holds for any algebraic system Σ such that its generic element is irreducible* (2).

COROLLARY 2. – *An algebraic system Σ of index $\nu = 1$ is either a linear system or a power of a linear system.*

[Because $\nu = 1$ if and only if $\rho = 1$ and the extension $K(Z) \supseteq \varphi^*(K(\mathbf{P}^r))$ is purely inseparable—omitting the suffix 1 for simplicity].

Remark. – Corollary 2 is a well known property of algebraic systems (see for instance [9] or [15]) to which we shall return at the end of Section IV.

Before giving the proof of Proposition 2 we need some additional information about 1-dimensional algebraic systems.

Let Σ be any 1-dimensional algebraic system, and introduce non-homogeneous coordinates $x = (x_1 \cdots, x_r)$ in $\mathbf{P}^r$ and the corresponding non-homo-

(2) So we have that the Index Theorem is true in particular in any characteristic for the *general* algebraic system Σ (when $r \geq 2$). Moreover, observe that in *Example 2* we have $\rho = 2^\delta > 1$.

geneous coordinates $a = (a_{k_1 \cdots k_r})$ in $\mathbf{P}^N$, so that the variety W is represented by the relation

$$\sum_{(k)} a_{k_1 \cdots k_r} x_1^{k_1} \cdots x_r^{k_r} + 1 = 0 \qquad \sum_{(k)} a_{(k)} x^{(k)} + 1 = 0 \,,$$

$$k_i \geq 0 \quad , \quad (k) \neq (0) \quad , \quad \sum_{i=1}^{r} k_i \leq n$$

(where we suppose that V is not contained in the «infinite hyperplane» defined by our choice of coordinates). Let $\eta = (\eta_{(k)})$ be a generic point of V over K, and write $f_\eta : \sum_{(k)} \eta_{(k)} x^{(k)} + 1 = 0$ as the *generic element* of Σ over K. Consider the factorization of $f_\eta(x)$ in $K(\eta)(x)$, write $f_\eta(x) = \varphi_1(x)^{m_1} \cdots \cdots \varphi_{\rho_0}(x)^{m_{\rho_0}}$, where $m_j \geq 1$, $j = 1, \cdots, \rho_0$, and $\varphi_1(x), \cdots, \varphi_{\rho_0}(x)$ are distinct prime elements of $K(\eta)[x]$.

LEMMA 2. – ρ_0 *is equal to* ρ, *and we can dispose of the suffix* j *in such a way that* $m_j = d_j$, $j = 1, \cdots, \rho$.

Proof. Suppose that $\Phi_\alpha(a) \in K[a]$, $\alpha = 1, \cdots, s$, generate the prime ideal of V over K. Then it is clear from the natural isomorphism

$$K[a, x] \Big/ \Big(\Phi_\alpha(a) \,,\, \sum_{(k)} a_{(k)} x^{(k)} + 1 \Big) \xrightarrow{\ \sim\ } K[\eta][x] \Big/ \Big(\sum_{(k)} \eta_{(k)} x^{(k)} + 1 \Big)$$

that there exists a 1—1 correspondence between the factors $\varphi_1(x), \cdots, \varphi_{\rho_0}(x)$ and the varieties $Z_1, \cdots, Z_\rho$ for which $\rho = \rho_0$. Then the identities $m_j = d_j$ follow from Lemma 1, because the cycle $(\mathbf{P}^r \times \eta) \cdot_{\mathbf{P}^r \times V} Z_j$ is *rational* over $K(\eta)$.

After Lemma 2, we can interpret the correspondence $l \mapsto j(l)$, $l = 1, \cdots, \nu$, $j(l) = 1, \cdots, \rho$, in Section II as follows: the point $\xi \times \eta_l$ belongs to the variety Z_j which corresponds to the unique factor $\varphi_j(x)$ for which $\varphi_j(\xi) = 0$ [it is clear from $f_\eta(\xi) = \varphi_1(\xi)^{d_1} \cdots \varphi_\rho(\xi)^{d_\rho} = 0$ that at least one of those factors must be 0 in ξ, and the unicity of such a factor follows from the hypothesis that ξ is a simple point of $|f_\eta|$].

Furthermore, it is useful to point out that from Lemma 2 it follows that the cycle Z is simple if and only if all the exponents m_j are equal to 1, and so that Z can be simple also in the case of variable multiple components. Accordingly, the equality $m = \nu$ can hold also for algebraic systems such that V is contained in $\mathfrak{V}$:

Example 3. – Suppose $p > 0$, and consider on the projective line the linear system Σ, depending on a parameter t, $x^p + t = 0$. Now we have $m = \nu = \rho = 1$, in spite of the fact that Σ has multiple variable components.

Of course a similar occurrence may no longer present itself if $p = 0$, because if $f_\eta(x)$ has no multiple components in $K(\eta)[x]$ then it has no multiple components in $\overline{K(\eta)}[x]$ either.

This remark provides the *converse* of the Index Theorem when $p = 0$ (and of course for any $\delta \geq 1$):

PROPOSITION 3. – *When* $p = 0$ *the equality* $m = \nu$ *holds if and only if* Σ *has no variable multiple components*.

This proposition has an obvious geometric interpretation in the space $\mathbf{P}^N$, furnishing a relation between the tangent spaces to the K–closed set $\mathfrak{Y}$ (in its simple points) and the hyperplanes L_x of the Veronese variety of indices (r, n). For instance, in the case $r = 1$, $n = 3$, $\mathfrak{Y}$ is a surface in $\mathbf{P}^3$ with a multiple curve, representing those cycles of type $3x$ in $\mathbf{P}^1$; the tangent plane to $\mathfrak{Y}$ in a simple point η, representing a cycle of the type $2x_1 + x_2$, $x_1 \neq x_2$, coincides with L_{x_1}.

Furthermore, one can observe that from Proposition 3 it follows a well known property of linear systems in a projective space, namely that a linear system—*without fixed components*—cannot have multiple variable components. This is a consequence of the 1st Theorem of Bertini (see for instance [9] or [18]), and the hypothesis $p = 0$ is now of course essential.

After these remarks, we proceed with the proof of Proposition 2.

Proof. It will be sufficient to prove the assertion in the case $\delta = 1$ (and $p > 0$). Let ξ be a generic point of $|f_\eta|$ over $K(\eta)$ (3) (so that ξ is also a generic point of $\mathbf{P}^r$ over K), and p^f be the inseparability degree of $K(Z)$ over $\varphi^*(K(\mathbf{P}^r))$—here omitting again the suffix 1 for simplicity—, i.e. of the extension $K(\xi, \eta) \supseteq K(\xi)$. Then we have that $K(\xi, \eta^{p^f})$ is separable over $K(\xi)$, whence the equality

$$\text{degree of } p^f\Sigma = \text{index of } p^f\Sigma$$

where $p^f\Sigma$ is the algebraic system defined by the generic element over K

$$\sum_{(k)} \eta_{(k)}^{p^f} x^{(k)p^f} + 1 = 0 \qquad \text{(with the notation as before).}$$

In fact it is clear that condition (ii) is fulfilled for $p^f\Sigma$ by construction. Moreover also the condition of simplicity of the cycle associated with $p^f\Sigma$ is fulfilled, in spite of the fact that $p^f\Sigma$ could have *a priori* multiple variable components (this however is not the case *a posteriori* since $p^f = 1$), because $\sum_{(k)} \eta_{(k)}^{p^f} x^{(k)p^f} + 1$ is irreducible in $K(\eta^{p^f})[x^{p^f}]$ and then also in $K(\eta^{p^f})[x]$; we can therefore use Lemma 2 in order to obtain the required conclusion (recall that necessarily $K(\eta^{p^f}) \neq K(\eta^{p^f})^p$, i.e. $K(\eta^{p^f})$ is not *perfect*).

Finally it is sufficient to observe that

$$\text{degree of } p^f\Sigma = m, \quad \text{index of } p^f\Sigma = \nu, \quad \text{to get } m = \nu, \qquad \text{q.e.d.}$$

We end this Section by recalling that the Index Theorem in the case $\nu = 1$ is used in the classical proof of *Luroth's Theorem* (see for instance [6], [9] or [11]), and remarking that Prop. 2 allows us to repeat the same kind of argument also in the case $p > 0$ (we get in this way a rather different proof, similar for instance to those given in [3] or [12] but in a slightly more general version).

(3) We mean of course a generic point of one of the irreducible components of $|f_\eta|$.

Proposition 4 (*Luroth's Theorem*). – *Let k be any field contained in Ω, $k(\xi) = k(\xi_1, \cdots, \xi_r)$ a pure transcendental extension of k in Ω of transcendence degree r, $k(\eta) = k(\eta_1, \cdots, \eta_s)$ any subfield of $k(\xi)$ of transcendence degree 1 over k. Then $k(\eta)$ is still a pure transcendental extension of k.*

Proof. Suppose that $\xi_1, \cdots, \xi_{r-1}$ is a transcendence basis of $k(\xi)$ over $k(\eta)$, i.e. that $k(\xi)$ is a simple algebraic extension of $k(\eta, \xi_1, \cdots, \xi_{r-1})$. Consider a minimal polynomial $H(x_r)$ of ξ_r over $k(\eta, \xi_1, \cdots, \xi_{r-1})$, which is a prime element of $k(\eta)[\xi_1, \cdots \xi_{r-1}][x_r]$. The corresponding prime polynomial $H(x) = H(x_1, \cdots, x_{r-1}, x_r) \in k(\eta)[x_1, \cdots, x_{r-1}, x_r] = k(\eta)[x]$ – under the natural isomorphism between these two rings – defines, as a generic element over k, a 1-dimensional algebraic system Σ of hypersurfaces in a projective space $\mathbf{P}^r$ (such that ξ is one of its generic points over k). By construction, Σ has the index $\nu = 1$; we claim also that the degree of Σ is 1. In fact, according to Lemma 2, it will be sufficient to prove that $H(x)$ is still prime as an element of $K(\eta)[x]$, where K is the algebraic closure of k in Ω; i.e., that $H(x_r)$ is still prime as an element of $K(\eta)[\xi_1, \cdots, \xi_{r-1}][x_r]$, namely $[K(\xi) : K(\eta, \xi_1, \cdots, \xi_{r-1})] = [k(\xi) : k(\eta, \xi_1, \cdots, \xi_{r-1})]$. The last equality holds because the extension $k(\xi) \supseteq k$ is *regular*, i.e. $k(\xi)$ and K are *linearly disjoint* over k, hence $k(\xi)$ and $K(\eta, \xi_1, \cdots, \xi_{r-1})$ are linearly disjoint over $k(\eta, \xi_1, \cdots$ $\cdots, \xi_{r-1})$ ([9], Prop. IV, p. 290), whence the conclusion ([9], Prop. I, p. 290). We have thus proved that there exist elements $\tau \in k(\eta)$, $F(x)$, $G(x) \in k[x]$ such that $H(x) = F(x) + \tau G(x)$. Then it is clear that

$$[k(\xi) : k(\eta, \xi_1, \cdots, \xi_{r-1})] = [k(\xi) : k(\tau, \xi_1, \cdots, \xi_{r-1})],$$

whence $k(\eta, \xi_1, \cdots, \xi_{r-1}) = k(\tau, \xi_1, \cdots, \xi_{r-1})$ [because $k(\eta, \xi_1, \ldots, \xi_{r-1}) \supseteq$ $\supseteq k(\tau, \xi_1, \cdots, \xi_{r-1})$]. The conclusion is now easy, since $k(\tau, \xi_1, \cdots, \xi_{r-1})$ and $k(\eta)$ are linearly disjoint over $k(\tau)$ and so it follows as before that

$$1 = [k(\eta, \xi_1, \ldots, \xi_{r-1}) : k(\tau, \xi_1, \cdots, \xi_{r-1})] = [k(\eta) : k(\tau)], \qquad \text{q.e.d.}$$

IV. – The Index Theorem for algebraic systems of hypersurfaces in a projective variety

Let $U \subset \mathbf{P}^r$ ($r \geq 2$) be a d-dimensional projective variety ($1 \leq d \leq r-1$) defined over the field K, and suppose for the sake of simplicity that U is *normal*. The (d$-$1)-dimensional positive cycles of $\mathbf{P}^r$ whose support is contained in U (we call them the *hypersurfaces* of U) and having a fixed degree μ are represented by the points of a K–closed subset $\mathrm{Ch}^{d-1,\mu}(U)$ of the *Chow scheme* (4) $\mathrm{Ch}^{d-1,\mu}(\mathbf{P}^r)$, representing all (d$-$1)-dimensional cycles of degree

(4) The word scheme is now generally used instead of the word variety because $\mathrm{Ch}^{d-1,\mu}(\mathbf{P}^r)$ may well be reducible. For all that concerns these preliminaries about algebraic systems of cycles look for instance at [1] or [7], in addition to the works already quoted in section I.

μ of $\mathbf{P}^r$. For each point η in $\mathrm{Ch}^{d-1,\mu}(U)$ we shall indicate by f_η the hypersurface of U corresponding to η, and call η the *Chow point* of f_η.

Let V be a δ-dimensional ($\delta \geq 1$) subvariety of $\mathrm{Ch}^{d-1,\mu}(U)$ defined over K; the set of all hypersurfaces of U corresponding to points of V will be called the δ-dimensional (irreducible) *algebraic system* associated with V, write $\Sigma = \Sigma(V)$ (5).

As in Section I, the *index* ν of Σ is defined as the number of distinct hypersurfaces of Σ which pass through δ generic points $\xi_1, \cdots, \xi_\delta$ of U over K (i.e., $\xi_1 \times \cdots \times \xi_\delta$ has to be a generic point of $U \times \cdots \times U$ over K).

From now on, suppose that Σ *is totally contained in a linear system* $\mathfrak{L}$ *of hypersurfaces of* U. There exists then some $(d-1)$-cycle g in U – not necessarily positive – such that $\mathfrak{L}+g$, the set of all cycles of U of the type $f+g$ where $f \in \mathfrak{L}$, is a set of hypersurfaces of U *cut out* on U by some linear system $\mathfrak{L}'$ of hypersurfaces of $\mathbf{P}^r$; i.e., no hypersurface of $\mathfrak{L}'$ contains U, and $f' \cdot_{\mathbf{P}^r} U$ describes $\mathfrak{L}+g$ while f' describes $\mathfrak{L}'$. Correspondingly, there exists some algebraic system $\Sigma' \subseteq \mathfrak{L}'$ which cuts out $\Sigma + g$ on U; we shall call it an algebraic system *associated* with Σ. Let ν' be the index of Σ' and m' its degree; the classical Index Theorem for algebraic systems of hypersurfaces of U asserts that $m' = \nu' = \nu$ provided that Σ, and hence also $\Sigma + g$ and Σ', have no variable multiple components (now we mean of course *with the possible exception of fixed components*). This theorem is usually proved when $p = 0$ by means of differential methods (see for instance [9]); we want to give here a complete algebraic proof of it in any characteristic. Needless to say, when $p > 0$ we shall need some further assumptions, for in this case the single hypothesis that Σ has no multiple variable components does not necessarily imply either $m' = \nu'$ or $\nu' = \nu$ even when $m' = \nu'$. Furthermore, as we shall show, such an algebraic system on U may be derived from distinct associated systems in $\mathbf{P}^r$ with different indices.

Example 4. – Take on the projective line $\mathbf{P}^1$ the algebraic system

$$\Sigma : x^2 - (t + t^p)\, x + t^{p+1} = 0$$

we have previously met (Example 2), and consider in the projective plane with non-homogeneous coordinates x, y the following two algebraic systems associated with Σ,

$$\Sigma' : x^2 - (t + t^p)\, x + t^{p+1} = 0,$$

$$\Sigma'' : x^2 + ty - (t + t^p)\, x + t^{p+1} = 0.$$

With obvious notation, it is clear that $\nu' = 2$, $m' = p + 1$ while $m'' = \nu'' = p + 1$.

Before giving the more general form of the Index Theorem, it is convenient to recall that we can get any s-dimensional ($s \geq 1$) linear system $\mathfrak{L}$ on

(5) Each of the systems we consider in this section may have fixed components.

U in the following way (FAG, IX, Theorem 14; in Section II we have used the same construction in a special case). Let $\mathbf{P}^s$ be the projective s–dimensional space associated with $\mathfrak{L}$ and φ be the corresponding rational map from U to the dual space $\overset{*}{\mathbf{P}}{}^s$ of $\mathbf{P}^s$ (the so-called *projective image* of $\mathfrak{L}$). Take a generic point ξ of U over K and a generic point η of the hyperplane $\varphi(\xi) \subset \mathbf{P}^s$ over $K(\xi)$, and then consider the $(d + s - 1)$–dimensional subvariety W of $U \times \mathbf{P}^s$ defined by the generic point $\xi \times \eta$ over K. One can easily prove that for any point $\eta \in \mathbf{P}^s$ the cycle $(U \times \eta) \cdot_{U \times \mathbf{P}^s} W$ is *defined* [i.e., $U \times \eta$ and W meet properly in U along any component of $(U \times \eta) \cap W$; actually, any such component is simple in $U \times \mathbf{P}^s$], and that $(U \times \eta) \cdot_{U \times \mathbf{P}^s} W = f_\eta \times \eta$ where $f_\eta = pr_U[(U \times \eta) \cdot_{U \times \mathbf{P}^s} W]$ is the cycle in $\mathfrak{L}$ corresponding to the point $\eta \in \mathbf{P}^s$ (we employ here the natural isomorphism between $\mathbf{P}^s$ and the subvariety representing $\mathfrak{L}$ in the suitable Chow scheme of U).

We are now in the position to tackle our problem. Given $\Sigma \subseteq \mathfrak{L}$, suppose that $\mathfrak{L}$ is defined as before by the irreducible cycle W in the product variety $U \times \mathbf{P}^s$; since Σ is contained in $\mathfrak{L}$, it is represented in $\mathbf{P}^s$ by some subvariety V. We can introduce the cycle $Z = W \cdot_{U \times \mathbf{P}^s} (U \times V)$, which has its support contained in $U \times V$, and we see as in Lemma 1 that $f_\eta = pr_U[(U \times \eta) \cdot_{U \times V} Z]$ for any simple point $\eta \in V$ (perhaps defined over an extension of K). Suppose now that $\delta = 1$ (for the same typographical reasons as in Section I), and call m the degree of V as a curve in $\mathbf{P}^s$ and $Z_1, \cdots, Z_\rho$ the irreducible components of Z such that their geometric projection over U is U itself (there may exist further irreducible components of Z, corresponding to the fixed components of Σ). After having then defined $\varphi_1, \cdots, \varphi_\rho$ as the restrictions of $pr_U : U \times V \to U$ to $Z_1, \cdots, Z_\rho$, we can prove:

PROPOSITION 4. – *With the notation as before, the equality $m = \nu$ holds provided that Σ has no variable multiple components and each morphism $\varphi_j : Z_j \to U$, $j = 1, \cdots, \rho$, is separable.*

Proof. One can repeat the same argument given for the Index Theorem (case $\delta = 1$) in Section II, with obvious modifications because the hypothesis that Σ has no variable multiple components implies that Z is a simple cycle and there exists a natural isomorphism from $\mathbf{P}^s$ into the suitable Chow scheme of U.

From Proposition 4 the new form of the Index Theorem follows easily, namely:

INDEX THEOREM (case $\delta = 1$). – *With the notation as before, let Σ' be any algebraic system in $\mathbf{P}^r$ associated with Σ. Then the identities $m' = \nu' = \nu$ hold, provided that*

(i) *Σ has no variable multiple components*;

(ii) *each morphism $\varphi_j : Z_j \to U$, $j = 1, \cdots, \rho$, is separable* (6).

(6) This condition has of course an intrinsic meaning with respect to the given system Σ.

Proof. Let Σ' be represented by a curve V' in the suitable projective space $\mathbf{P}^{N'}$, and $\mathfrak{L}'$ by a linear subspace $\mathbf{P}'^{s}$ of $\mathbf{P}^{N'}$. Then there exists a natural K-linear isomorphism between $\mathbf{P}^{s}$ and $\mathbf{P}'^{s}$, in which V corresponds to V', and thus it follows from Prop. 4 that $m' = \nu$, q.e.d. on account of the inequalities $m' \geq \nu' \geq \nu$.

Remark. – A simple alternative proof can now be given when $p = 0$. Let us introduce, with the same notation as in Section II, the variety W' in $\mathbf{P}^{r} \times \mathbf{P}^{N'}$, the cycle $Z' = \sum_{j=1}^{\rho'} Z'_j = W' \cdot_{\mathbf{P}^r \times \mathbf{P}^{N'}} (\mathbf{P}^r \times V')$ and the maps $\varphi'_j : Z'_j \to \mathbf{P}^r$, $j = 1, \cdots, \rho'$, induced by $pr_{\mathbf{P}^r} : \mathbf{P}^r \times \mathbf{P}^{N'} \to \mathbf{P}^r$ (7). For any $j = 1, \cdots, \rho'$, we define the K-closed subset Δ'_j of $\mathbf{P}^r$ as the locus of the points $x \in \mathbf{P}^r$ such that

$$\#[(x \times \mathbf{P}^{N'}) \cdot_{\mathbf{P}^r \times \mathbf{P}^{N'}} Z'_j] < [K(Z'_j) : \varphi'^{*}_j (K(\mathbf{P}^r))]$$

(Δ'_j is the so-called *discriminant* of φ'_j). $\Delta' = \bigcup_{j=1}^{\rho'} \Delta'_j$ is a pure K-closed subset of $\mathbf{P}^r$ of codimension 1; and it is clear that $\nu < \nu'$ if and only if U is contained in Δ'. Now let ξ be a generic point of U over K, and η be a point of V' such that $i(L_\xi \cdot V', \eta ; \mathbf{P}^{N'}) > 1$; one can easily check that the hypersurface of $\mathbf{P}^r$ corresponding to η has the same tangent hyperplane (8) as Δ' at ξ, and thus Σ' cuts out on U a system with variable multiple components.

It is straightforward to extend the above statement to the case $\delta > 1$, as we have done in Section II. We have just to consider the cycle $\underbrace{Z \times \cdots \times Z}_{\delta \text{ times}}$ in $\underbrace{(U \times V) \times \cdots \times (U \times V)}_{\delta \text{ times}}$, identify $(U \times V) \times \cdots \times (U \times V)$ with $U \times \cdots \times U \times \times V \times \cdots \times V$, and then introduce in the last variety the subvariety $U \times \cdots \cdots \times U \times \Delta_V$, where Δ_V is the *diagonal* subvariety of $V \times \cdots \times V$, and the cycle $Z \times \cdots \times Z \cdot_{U \times \cdots \times U \times V \times \cdots \times V} U \times \cdots \times U \times \Delta_V$. We shall indicate this cycle again by Z and identify $U \times \cdots \times U \times \Delta_V$ with $U \times \cdots \times U \times V$; then, if $Z_1, \cdots, Z_\rho$ are the irreducible components of Z having the geometric projection $U \times \cdots \times U$ under $pr_{U \times \cdots \times U} : U \times \cdots \times U \times V \to U \times \cdots \times U$, we introduce the finite morphism $\varphi_j : Z_j \to U \times \cdots \times U$, $j = 1, \cdots, \rho$, induced by the projection map $pr_{U \times \cdots \times U}$, and we finally prove:

INDEX THEOREM (General case). – *With the notation as before, the identities* $m' = \nu' = \nu$ *hold, provided that*

(i) Σ *has no variable multiple components,*

(ii) *each morphism* $\varphi_j : Z_j \to U \times \cdots \times U$, $j = 1, \cdots, \rho$ *is separable.*

It would be easy to extend to the general case of an algebraic system of hypersurfaces on U the considerations made in Section III, for instance Proposition 2 and its two corollaries, but we do not insist on this point. We only

(7) Suppose for the sake of simplicity that Σ' has no fixed components.

(8) Possibly undetermined, i.e. ξ is a singular point of that hypersurface.

wish to point out that the hypothesis that Σ is totally contained in a linear system is of course essential throughout the present Section, even for the validity of the last of the two corollaries just recalled. In fact it is well known that there exist some algebraic surfaces (see for instance [4], [9], [14]) carrying an *irrational pencil* (i.e., an algebraic non-linear system of curves of index 1), the generic element of which is irreducible.

Remark. – A Theorem of Enriques ([4]) asserts that any δ–dimensional algebraic system on U with index 1, the generic element of which is irreducible, is necessarily linear whenever $\delta \geq 2$. The two hypotheses are now both essential, because one can find for any $\delta \geq 2$ δ–dimensional algebraic non-linear systems of index 1 free from variable multiple components.

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