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Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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# RENDICONTI

DELLE SEDUTE

## DELLA ACCADEMIA NAZIONALE DEI LINCEI

**Classe di Scienze fisiche, matematiche e naturali**

*Seduta del 15 dicembre 1973*

*Presiede il Presidente della Classe* BENIAMINO SEGRE

## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Matematica.** — *Complementarity between nilpotent selfmappings and periodic autohomeomorphisms.* Nota (*) di LUDVIK JANOS, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia $(X, f)$ una coppia formata da uno spazio di Hausdorff compatto e da una trasformazione continua $f: X \to X$ tale che per qualche $n \geq 1$ l'iterata $f^n$ è idempotente, ossia, $f^{2n} = f^n$. Si mostra che la categoria C di tali coppie può essere immessa naturalmente e fedelmente nel prodotto $C_1 \times C_2$ delle due sotto–categorie piene $C_1$ e $C_2$ dove $C_1$ consiste delle coppie nilpotenti ($f^n$ è costante per qualche $n \geq 1$) e $C_2$ degli autoomeomorfismi periodici ($f^n$ è l'identità per qualche $n \geq 1$).

### 1. INTRODUCTION

Let X be a compact Hausdorff space (all spaces considered here are compact Hausdorff) and $f: X \to X$ a continuous selfmapping of X. Considering $f$ as an element of the topological semigroup $X^X$ of all continuous selfmappings of X with respect to functional composition and compact open topology, we denote by $\Gamma(f)$ the closed subsemigroup of $X^X$ generated by $f$. This semigroup has been thoroughly investigated by A. D. Wallace ([1] and [2]) who obtained the following important result, (Swelling Lemma), concerning those selfmappings for which $\Gamma(f)$ is compact:

THEOREM 1.1. (A. D. Wallace). *Let* X *be a compact Hausdorff space and* $f: X \to X$ *a continuous selfmapping such that* $\Gamma(f)$ *is compact. Denoting by* A *the intersection of all iterates* $f^n(X)$, *i.e.*, $A = \cap \{f^n(X) \mid n \geq 1\}$ *the following statements hold*:

(i) *The restriction* $f \mid A$ *of* $f$ *to* A *is a homeomorphism of* A *onto itself.*

(ii) *There exists a unique idempotent* $r \in \Gamma(f)$ *which is a retraction of* X *onto* A.

(*) Pervenuta all'Accademia il 22 settembre 1973.

We shall apply this theorem to the case where $\Gamma(f)$ is finite. In this case evidently the existing idempotent $r$ is an iteration $f^n$ of $f$ for some $n \geq 1$. Let C denote the category whose objects Obj (C) consist of such pairs and whose morphisms $\varphi \in$ Morph $[(X, f), (Y, g)]$ are such continuous mappings $\varphi : X \rightarrow Y$ for which $g \circ \varphi = \varphi \circ f$ for $(X, f)$ and $(Y, g) \in$ Obj (C).

We denote by $C_1$ the full subcategory of C generated by nilpotent pairs ($f$ is such that $f^n$ is constant for some $n \geq 1$) and by $C_2$ the full subcategory generated by pairs $(X, f)$ where $f$ is a periodic autohomeomorphism ($f^n$ is the identity mapping for some $n \geq 1$).

Using Theorem 1.1. we shall construct functors $F_1 : C \rightarrow C_1$ and $F_2 : C \rightarrow C_2$ and using our previous results [3] we shall prove our main result:

THEOREM 1.2. *The product functor* $F = F_1 \times F_2$ *provides a faithful embedding of the category* C *into the product* $C_1 \times C_2$.

*Thus in this sense the subcategories* $C_1$ *and* $C_2$ *can be viewed as complementary in the category* C.

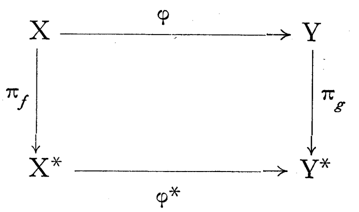
## 2. CONSTRUCTION OF FUNCTORS $F_1$ AND $F_2$

Let $(X, f) \in$ Obj (C), we consider the relation R on X defined by $R = A \times A \cup \{(x, x) \mid x \in X\}$ with A defined as in Theorem 1.1., and consider $X^* = X/R$. Thus the space $X^*$ is obtained by shrinking A to a point and is obviously again compact Hausdorff. Observing that there is a unique continuous mapping $f^* : X^* \rightarrow X^*$ rendering the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X^* & \xrightarrow[\quad f^* \quad]{} & X^* \end{array} \qquad \text{commutative}$$

($\pi$ being the natural projection), we obtain in this natural way a new pair $(X^*, f^*)$ which is evidently nilpotent since $f^{*n}$ takes $X^*$ to a point if $f^n$ takes X onto A.

On the other hand Theorem 1.1. says that the restriction $f \mid A$ is a homeomorphism of A onto itself which in our case is evidently periodic. Denoting A by $X^{**}$ and $f \mid A$ by $f^{**}$ we just found two objects, $(X^*, f^*) \in$ Obj $(C_1)$ and $(X^{**}, f^{**}) \in$ Obj $(C_2)$ assigned in natural way to the object $(X, f) \in$ Obj (C). These objects are the values on Obj (C) of the functors $F_1$ and $F_2$ to be defined now. In order to extend their definition to morphisms of C let $(Y, g)$ be another object in C and $\varphi : (X, f) \rightarrow (Y, g)$ a morphism from $(X, f)$ to $(Y, g)$ and let us denote by $\varphi^*$ the mapping from $X^*$ into $Y^*$ induced by $\varphi$, i.e. the mapping $\varphi^* : X^* \to Y^*$ for which the diagram:



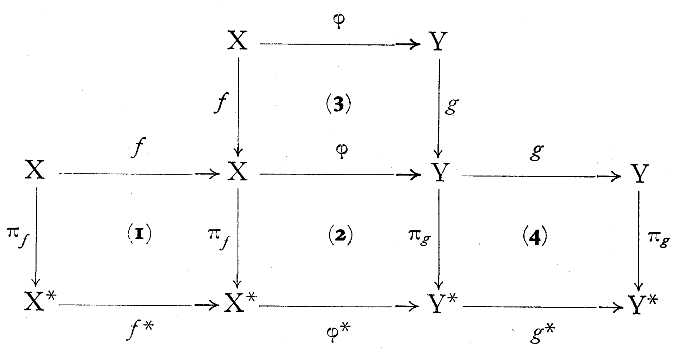

is commutative,

where $\pi_f$ and $\pi_g$ are the natural projections associated with the pairs $(X, f)$ and $(Y, g)$.

In order to prove that $\varphi^*$ is a morphism we need the following.

LEMMA 2.1. *Let* $(X, f)$ *and* $(Y, g)$ *be objects in* C *and* $\varphi : (X, f) \to (Y, g)$ *a morphism in* C. *Then the mapping* $\varphi^*$ *constructed above is a morphism, i.e.* $\varphi^* : (X^*, f^*) \to (Y^*, g^*)$.

*Proof.* We have to show that $g^* \circ \varphi = \varphi^* \circ f^*$. To this end observe the following four commutative diagrams labelled by (1), (2), (3), (4) as indicated:




Let $x^* \in X^*$, then there is $x \in X$ such that $\pi_f(x) = x^*$ and by (1) we have $f^*(x^*) = f \circ \pi_f(x) = \pi_f \circ f(x)$. Applying $\varphi^*$ we have by (2) $\varphi^* \circ f^*(x^*) = \varphi^* \circ \pi_f \circ f(x) = \pi_g \circ \varphi \circ f(x)$ and by (3) we have $\pi_g \circ \varphi \circ f(x) = \pi_g \circ g \circ \varphi(x)$ and finally by (4) we get $\pi_g \circ g \circ \varphi(x) = g^* \circ \pi_g \circ \varphi(x)$ which by (2) equals $g^* \circ \varphi^*(x^*)$. Thus $\varphi^* \circ f^*(x^*) = g^* \circ \varphi^*(x^*)$ q.e.d.

The functor $F_1 : C \to C_1$ is now defined simply by putting $F_1(X, f) = (X^*, f^*)$ and $F(\varphi) = \varphi^*$ for $(X, f), (Y, g) \in \mathrm{Obj}(C)$ and $\varphi \in \mathrm{Morph}[(X, f), (Y, g)]$.

Similarly in order to define the functor $F_2$ we need the following.

LEMMA 2.2. *With* $(X, f), (Y, g)$ *and* $\varphi$ *as in Lemma 2.1. we have* $\varphi(X^{**}) \subset Y^{**}$ *where the meaning of* $X^{**}$ *and* $Y^{**}$ *was defined above.*

*Proof.* Let $x \in X^{**} = \cap \{f^n(X) \mid n \geq 1\}$. Since $\varphi \circ f = g \circ \varphi$ it follows that $\varphi \circ f^n = g^n \circ \varphi$ for every $n \geq 1$. Since $f \mid X^{**}$ is a homeomorphism onto, there exists for every $n \geq 1$ an element $x_1 \in X^{**}$ such that $x = f^n(x_1)$. Applying $\varphi$ we get $\varphi(x) = \varphi \circ f^n(x_1) = g^n \circ \varphi(x_1) \in g^n(Y)$. Thus, since $n$ is arbitrary we conclude that $\varphi(x) \in Y^{**}$, q.e.d.

This lemma finally shows that the assignment $F_2(X, f) = (X^{**}, f^{**})$ and $F_2(\varphi) = \varphi^{**} = \varphi \mid X^{**}$ is a functor from the category C into the category $C_2$.

Having defined the functors $F_1 : C \to C_1$ and $F_2 : C \to C_2$ we observe the trivial fact that both are onto (projecting the category C onto the subcategories $C_1$ and $C_2$ respectively). Next we shall show that the product functor $F = F_1 \times F_2 : C \to C_1 \times C_2$ which takes the category C into the cartesian product $C_1 \times C_2$ is faithful. To achieve this we define the functor $P : C_1 \times C_2 \to C$ by putting $P[(X_1, f_1), (X_2, f_2)] = [(X_1 \times X_2), f_1 \times f_2]$ and $P[\varphi_1, \varphi_2] = \varphi_1 \times \varphi_2$ where $(X_1, f_1), (Y_1, g_1) \in \mathrm{Obj}(C_1)$ $(X_2, f_2), (Y_2, g_2) \in \mathrm{Obj}(C_2)$ and $\varphi_1 : (X_1, f_1) \to (Y_1, g_1)$ $\varphi_2 : (X_2, f_2) \to (Y_2, g_2)$.

The functor P is simply the cartesian product of pairs and morphisms between them. It is easy to verify that its values are in C. Considering the composite functor $S = P \circ F : C \to C$ we shall exhibit a natural transformation $\tau : I_C \to S$ from the identity functor $I_C$ to the functor S, and using our result [3] we will show that this natural transformation $\tau$ which assigns to each object $(X, f)$ in C a morphism $\tau(X, f)$ from $(X, f)$ to $S(X, f) = (X^* \times X^{**}, f^* \times f^{**})$, provides a topological embedding $\tau : X \to X^* \times X^{**}$. This means that two morphisms $\varphi$ and $\psi$ which are distinct and going from $(X, f)$ remain distinct when transformed under S. But this means that S is faithful and a fortiori F itself is faithful.

## 3. The natural transformation from the identity functor to S

LEMMA 3.1. *Let $(X, f)$ and $(Y, g)$ be objects in* C *and $\varphi$ a morphism from $(X, f)$ to $(Y, g)$. Denoting by $r_f$ and $r_g$ the corresponding idempotents in $\Gamma(f)$ and $\Gamma(g)$ respectively we claim that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\;r_f\;} & X \\ \varphi \downarrow & & \downarrow \varphi \\ Y & \xrightarrow[\;r_g\;]{} & Y \end{array}$$

*is commutative.*

*Proof.* We know that $r_f = f^n$ and $r_g = g^m$ for some $n$ and $m \geq 1$. Since $r_f^2 = r_f$ and $r_g^2 = r_g$ we can write $r_f = f^k$, $r_g = g^k$ where $k = mm$, and since $g \circ \varphi = \varphi \circ f$ our assertion follows.

Now we give the promised definition of the natural transformation $\tau : I_C \to S$.

If $(X, f)$ is an object in C, then $S(X, f) = P \circ F(X, f)$ is the pair $(X^* \times X^{**}, f^* \times f^{**})$. Defining $\tau : X \to X^* \times X^{**}$ by $\tau(x) = (\pi_f(x), r_f(x))$ where $\pi_f$ is the natural projection $\pi_f : X \to X^*$ and $r_f$ the abovementioned idempotent element in $\Gamma(f)$, the Theorem 2.1. of the paper [3] says that $\tau$ is a topological embedding of X into $X^* \times X^{**}$ and at the same time a morphism from $(X, f)$ to $S(X, f)$. To show that $\tau$ is a natural transformation it remains to verify that for any $(X, f), (Y, g) \in \mathrm{Obj}(C)$ and any $\varphi : (X, f) \to (Y, g)$ the following diagram commutes:

$$\begin{array}{ccc} (X, f) & \xrightarrow{\tau(X, f)} & S(X, f) \\ \varphi \downarrow & & \downarrow S(\varphi) \\ (Y, g) & \xrightarrow{\tau(Y, g)} & S(Y, g) \end{array}$$

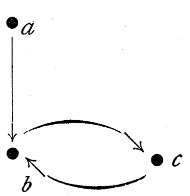
But this diagram, written in explicit form is

$$\begin{array}{ccc} (X, f) & \xrightarrow{\tau(X, f)} & (X^* \times X^{**}, f^* \times f^{**}) \\ \varphi \downarrow & & \downarrow \varphi^* \times \varphi^{**} \\ (Y, f) & \xrightarrow[\tau(Y, g)]{} & (Y^* \times Y^{**}, g^* \times g^{**}) \end{array}$$

and its commutativity follows easily from Lemma 3.1.

From what has been said at the end of section 2, this implies that F is faithful, proving thus our Theorem 1.2.

*Remark.* The functor $F : C \to C_1 \times C_2$ is not full. To show it, consider the finite pair $(X, f) \in \mathrm{Obj}(C)$ consisting of three objects, say $\{a, b, c\}$ and the selfmapping $f$ defined by arrows as follows:



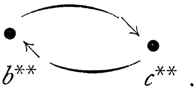

$$f = \begin{pmatrix} a, b, c \\ b, c, b \end{pmatrix}.$$

There are exactly three morphisms from $(X, f)$ to $(X, f)$, namely the identity $\varphi$ and $\psi$ defined by:

$$\varphi = \begin{pmatrix} a, b, c \\ b, c, b \end{pmatrix} \qquad \psi = \begin{pmatrix} a, b, c \\ c, b, c \end{pmatrix}.$$

The value of $F_1$ on $(X, f)$ can be similarly represented by the graph


and the value of $F_2$ on $(X, f)$ by the graph




Since the number of morphisms in $C_1$ going from $F_1(X, f)$ into itself is two (one except the identity) and the number of morphisms in $C_2$ going from $F_2(X, f)$ into itself is also two (also one except the identity) it follows that the total number of morphisms in $C_1 \times C_2$ going from $F_1 \times F_2(X, f)$ into itself is four, showing that the functor $F = F_1 \times F_2$ is not full.

## References

[1] A. D. Wallace, *Inverses in Euclidean mobs*, «Math. J. Okayama Univ.», *3*, 23–28, MR 15, 933 (1953).

[2] A. D. Wallace, *The Gebietstreue in semigroups*, «Nederl. Akad. Wetensch. Pros. Ser. A 59 Indag. Math.», *18*, 271–274, MR 18, 14 (1956).

[3] L. Janos, *On representations of selfmappings*, «Proceedings of A.M.S.», 529–533, November 1970.