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On a new method for studying the stability of Volterra integral equations

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Equazioni integrali. — *On a new method for studying the stability of Volterra integral equations.* Nota di ATHANASSIOS G. KARTSATOS e WILLIAM R. ZIGLER, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Mediante l'uso di teoremi di punto fisso si deduce la stabilità delle soluzioni di un'equazione di Volterra non lineare dalla stabilità delle soluzioni di una famiglia di equazioni di Volterra lineari.

1. INTRODUCTION

Consider the Volterra integral system

$$(*) \qquad x(t) = H(t) + \int_0^t K(t, s, x(s))\, x(s)\, ds,$$

where x, H are n–vectors and K is an $n \times n$ matrix. To the system (*) we can naturally associate the systems

$$(*)_u \qquad x(t) = H(t) + \int_0^t K(t, s, u(s))\, x(s)\, ds,$$

which are *linear* in x for every $u \in S$, S being a suitable subset of a Banach function space. Now it is natural to ask whether information about the solutions of the systems $(*)_u$, $u \in S$ can be used in order to establish corresponding results for the nonlinear system (*). This idea has been exploited by the first of the authors in the papers [2], [3], where bounded solutions were obtained under boundedness assumptions on the Systems $(*)_u$, $u \in S$. Our main intention here is to employ this method in order to obtain stability of the solution of (*). The passage from $(*)_u$ to (*) can be accomplished by one of the well known fixed point theorems.

The results of this paper have points of contact with the papers in the references, and the papers referred to therein.

2. PRELIMINARIES

In what follows, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $\|x\| = \sup_i |x_i|$ for each $x \in R^n$, $\|A\| = \sup_i \sum_k |a_{ik}|$ for every $n \times n$ real matrix $A = [a_{ij}]$, and $S_c = \{x \in R^n ; \|x\| \leq c\}$, where c is a positive number.

(*) Nella seduta del 15 dicembre 1973.

We also let $E_a = \{(t, s); 0 \leq s \leq t \leq a\}$ for $a > 0$ and $E_\infty = \{(t, s); 0 \leq s \leq t < \infty\}$. By $C[A, B]$ we mean the set of all continuous R^n-valued functions defined on the set A with values in the set B. If I is an interval, by $C_b[I, R^n]$ we mean $C[I, R^n]$ in case I is closed, and the space of all continuous and bounded R^n-valued functions on I, in case I is infinite. In both cases, $\|\cdot\|_I$ will denote the sup-norm of $C[I, R^n]$, under which it becomes a Banach space.

The following Lemma has been shown by Kartsatos [3].

LEMMA A. *For the equation*

$$x(t) = H(t, x(t)) + \int_0^t K(t, s, x(s))\, ds \tag{2.1}$$

assume the following:

(i) $H \in C[R_+ \times S_c, R^n]$ *and, for each* $m = 1, 2, \ldots$,

$$\|H(t, u_1) - H(t, u_2)\| \leq h_m \|u_1 - u_2\|,$$

for any $t \in [0, m]$, $u_1, u_2 \in S_c$, *where* h_m *is a constant with* $0 < h_m < 1$;

(ii) $K \in C[E_\infty \times S_c, R^n]$;

(iii) *there exists a sequence of functions* $x_m(t)$, $m = 1, 2, \ldots$, *such that, for each* m, $x_m(t)$ *is a solution of* (2.1) *on* $[0, m]$ *and* $\|x_m(t)\| \leq c$, $t \in [0, m]$.

Then there exists at least one solution $x(t)$ *of* (2.1), *which is defined on* R_+ *and satisfies* $\|x(t)\| \leq c$, $t \in R_+$.

In the following definition, it will be assumed that H, K in (*), $(*)_u$ satisfy the following

Hypothesis (Q): $H \in C[R_+, R^n]$; $K(t, s, v)$ *is an* $n \times n$ *matrix defined and continuous on the set* $E_\infty \times R^n$.

Under Hypothesis (Q), the linear system $(*)_u$ has a unique solution $x_u(t)$, $t \in R_+$ for any $u \in C[R_+, R^n]$.

DEFINITION 2.1 *The solution* $x(t)$, $t \in R_+$ *of* (*) *is said to be stable, if for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for each* $\bar{H} \in C[R_+, R^n]$ *with* $\|\bar{H} - H\|_{R_+} < \delta(\varepsilon)$ *there exists at least one solution* $\bar{x}(t)$, $t \in R_+$ *of the equation*

$$\bar{x}(t) = \bar{H}(t) + \int_0^t K(t, s, \bar{x}(s))\, \bar{x}(s)\, ds$$

with $\|\bar{x} - x\|_{R_+} \leq \varepsilon$.

DEFINITION 2.2. *The linear systems* $(*)_u$ *are said to be iso-stable if for every* $\varepsilon > 0$; *there exists* $\delta(\varepsilon) > 0$ *such that for every* $u \in C_b[R_+, R^n]$ *with* $\|u\|_{R_+} \leq \varepsilon$ *and every* $H \in C_b[R_+, R^n]$ *with* $\|H\|_{R_+} \leq \delta(\varepsilon)$ *the solution* $x_u(t)$, $t \in R_+$ *of* $(*)_u$ *satisfies* $\|x_u\|_{R_+} \leq \varepsilon$.

The reader should note that the number $\delta(\varepsilon)$ in Definition 2.2 does not depend on the particular function u.

3. Main Results

THEOREM 3.1. *Assume that the real* $n \times n$ *matrix* $K(t, s, v)$ *is defined and continuous on* $E_\infty \times R^n$. *Let the systems* $(*)_u$ *be iso-stable. Then the zero solution of* $(*)$ *(corresponding to* $H(t) \equiv 0$*) is stable in the sense of Definition 2.1.*

Proof. Fix $\varepsilon > 0$ and choose $\delta(\varepsilon) > 0$ to satisfy Definition 2.2. Let $H \in C[R_+, R^n]$ with $\|H\|_{R_+} \leq \delta(\varepsilon)$ be fixed, $I_1 = [0, 1]$, and consider the set $S = \{u \in C[I_1, R^n]; \|u\|_{I_1} \leq \varepsilon\}$. Then S is a closed, convex and bounded subset of the Banach space $C[I_1, R^n]$. We define the operator $T: S \to C[I_1, R^n]$ by letting Tu be the solution of $(*)_u$ restricted to the interval I_1. Since the systems $(*)_u$ are assumed iso-stable, and since $\|H\|_{R_+} \leq \varepsilon$, it follows that $\|x_u\|_{I_1} \leq \varepsilon$. Consequently $TS \subseteq S$. Furthermore, since $E_1 \times S_\varepsilon$ is a compact set in $R \times R \times R^n$, and K is continuous, we have that K is uniformly continuous on $E_1 \times S_\varepsilon$ and

$$\sup_{\substack{(t,s) \in E_1 \\ \|v\| \leq \varepsilon}} \|K(t, s, v)\| = p < \infty .$$

Now let $t_1, t_2 \in I_1$, $u \in S$, and $x = Tu$. Then we obtain

$$\|x(t_1) - x(t_2)\| \leq \|H(t_1) - H(t_2)\| + \left\| \int_{t_1}^{t_2} K(t, s, u(s))\, x(s)\, ds \right\| +$$

$$+ \left\| \int_0^t [K(t_1, s, u(s)) - K(t_2, s, u(s))]\, x(s)\, ds \right\| \leq \|H(t_1) - H(t_2)\| +$$

$$+ p\varepsilon\, |t_1 - t_2| + \varepsilon \sup_{\|u\| \leq \varepsilon} \int_0^t \|K(t_1, s, u) - K(t_2, s, u)\|\, ds ,$$

which shows the equicontinuity of the set TS.

Since TS is also bounded, it is a relatively compact subset of $C[I_1, R^n]$. Now we show that T is continuous. Let $\{u_m\}_{m=1}^\infty$ be a sequence in S such that

$$\lim_{m \to \infty} \|u_m - u\|_{I_1} = 0$$

where $u \in S$. Then, by what we showed above, $\{Tu_m\}_{m=1}^\infty$ is equicontinuous and uniformly bounded so that there exists a subsequence $\{Tu_{m_k}\}_{k=1}^\infty$ and $y \in S$ such that

$$\lim_{k \to \infty} \|Tu_{m_k} - y\|_{I_1} = 0 .$$

If we let $y_k = Tu_{m_k}$, then for each $k = 1, 2, \cdots$ we have

$$y_k(t) = H(t) + \int_0^t K(t, s, u_{m_k}(s))\, y_k(s)\, ds$$

and

$$(\mathrm{T}u)(t) = \mathrm{H}(t) + \int_0^t \mathrm{K}(t, s, u(s))\,(\mathrm{T}u)(s)\,\mathrm{d}s.$$

Thus,

$$\|y_k(t) - (\mathrm{T}u)(t)\|$$

$$\leq \int_0^t \|\mathrm{K}(t, s, u_{m_k}(s))\, y_k(s) - \mathrm{K}(t, s, u(s))\,(\mathrm{T}u)(s)\|\,\mathrm{d}s$$

$$\leq \int_0^t \|\mathrm{K}(t, s, u_{m_k}(s))\|\,\|y_k(s) - (\mathrm{T}u)(s)\|\,\mathrm{d}s$$

$$+ \int_0^t \|\mathrm{K}(t, s, u_{m_k}(s)) - \mathrm{K}(t, s, u(s))\|\,\|(\mathrm{T}u)(s)\|\,\mathrm{d}s$$

$$\leq p \int_0^t \|y_k(s) - (\mathrm{T}u)(s)\|\,\mathrm{d}s$$

$$+ \varepsilon \int_0^t \|\mathrm{K}(t, s, u_{m_k}(s)) - \mathrm{K}(t, s, u(s))\|\,\mathrm{d}s.$$

Now let $\eta > 0$ be given. Since $\mathrm{K}(t, s, v)$ is uniformly continuous on $\mathrm{E}_1 \times \mathrm{S}_\varepsilon$, there exists $\lambda > 0$ such that

$$\sup_{(t,s)\in \mathrm{E}_1} \|\mathrm{K}(t, s, u_1) - \mathrm{K}(t, s, u_2)\| < \eta$$

whenever $\|u_1 - u_2\| < \lambda$.

Since $u_{m_k} \to u$ uniformly on I_1, there exists an integer $k_0 > 0$ such that

$$\|u_{m_k}(s) - u(s)\| < \lambda$$

for all $s \in \mathrm{I}_1$ and all $k \geq k_0$. Consequently, if $k \geq k_0$ we have

$$\int_0^t \|\mathrm{K}(t, s, u_{m_k}(s)) - \mathrm{K}(t, s, u(s))\|\,\mathrm{d}s \leq \eta t \leq \eta,$$

from which it follows that for $k \geq k_0$

$$\|y_k(t) - (\mathrm{T}u)(t)\| \leq \varepsilon\eta + p \int_0^t \|y_k(s) - (\mathrm{T}u)(s)\|\,\mathrm{d}s.$$

Gronwall's inequality implies now that

$$\|y_k - \mathrm{T}u\|_{\mathrm{I}_1} \leq \varepsilon\eta e^p, \qquad k \geq k_0.$$

Thus $y_k \to Tf$ in the norm $\|\cdot\|_{I_1}$. Since we could equally well have started with any subsequence of $\{Tu_m\}_{m=1}^{\infty}$ rather than $\{Tu_m\}_{m=1}^{\infty}$ itself, we have shown that every subsequence of $\{Tu_m\}_{m=1}^{\infty}$ contains a subsequence converging to Tu in $C[I_1, R^n]$. Therefore, Tu_m converges uniformly in $C[I_1, R^n]$.

It now follows from Tychonov's theorem that the operator T has a fixed point in S. Obviously, such a fixed point is a solution of (*) on the interval $[0, 1]$. Now consider the interval $I_m = [0, m]$, where m is a positive integer. By repeating the above argument we obtain a sequence of functions $\{x_m(t)\}_{m=1}^{\infty}$ such that $x_m(t)$ is a solution of (*) on I_m and $\|x_m(t)\| \leq \varepsilon$, $t \in I_m$, $m = 1, 2, \cdots$ It now follows from Lemma A that there exists at least one solution $x(t)$ of (*) on R_+, which satisfies $\|x(t)\| \leq \varepsilon$, $t \in R_+$ and the proof is completed.

The above result can be applied to study the stability of systems having the more general form

$$(**) \qquad x(t) = H(t) + \int_0^t K(t, s, x(s))\, ds.$$

where the n-vector $K(t, s, u)$ is defined and continuous on $E_\infty \times R^n$, and, moreover, it is continuously differentiable with respect to u. Let $x(t)$, $t \in R_+$ be a fixed solution of (**) and let $\bar{x}(t)$, $t \in R_+$ satisfy

$$(3.1) \qquad \bar{x}(t) = H_1(t) + \int_0^t K(t, s, \bar{x}(s))\, ds.$$

Then if $y(t) \equiv \bar{x}(t) - x(t)$, $t \in R_+$, we have

$$(3.2) \qquad K(t, s, x(s)) - K(t, s, \bar{x}(s)) = \bar{K}(t, s, x(s), y(s))\, y(s),$$

where $\bar{K}(t, s, u, v)$ is an $n \times n$ matrix defined and continuous on $E_\infty \times R^n \times R^n$. For a proof of this statement the reader is referred to Lakshmikantham and Leela [1, p. 73]. Consequently, we obtain

$$(3.3) \qquad y(t) = [H_1(t) - H(t)] + \int_0^t \bar{K}(t, s, x(s), y(s))\, y(s)\, ds.$$

Since this is the form of (*), the stability of the solution $x(t)$ of (**) (Definition 2.1 holds also for (**)) can be studied through the stability of the zero solution (corresponding to $H_1 - H \equiv 0$) of the above system. In particular, we have the following

COROLLARY 3.1. *If the systems*

$$(3.3)_u \qquad y(t) = \theta(t) + \int_0^t \bar{K}(t, s, x(s), u(s))\, y(s)\, ds$$

are iso-stable, then the solution $x(t)$ *of* (*, *) *is stable in the sense of Definition 2.1 (where* (*) *is replaced by* (**)).

It is evident that analogous results hold if one considers instead of the space $C_b[R_+, R^n]$ for the function $H(t)$ in $(^*)_u$ other normed spaces of continuous functions. For example, the space LC_b of all $f \in C_b$ such that

$$\|f\|_L = \int_0^\infty \|f(t)\| \, dt < \infty.$$

Here the norm is

$$\|f\|_1 = \|f\|_{R_+} + \|f\|_L .$$

Now assume that the kernel $K(t, s, u)$ in $(^*)$ is defined and continuous on $E_\infty \times R^n$ and that the function $H(t)$, $t \in R_+$ belongs to LC_b. Then for each $u \in C_b[R_+, R^n]$ there exists a matrix $R_u(t, s)$ defined and continuous on E_∞ and such that

$$(3.4)_u \qquad R_u(t, s) = -K(t, s, u(s)) + \int_s^t K(t, \tau, u(\tau)) \, R_u(\tau, s) \, d\tau$$

for every $t \geq s \geq 0$. $R_u(t, s)$ is the resolvent kernel associated with $(^*)_u$, whose unique solution. $x_u(t)$, $t \in R_+$ can be written as

$$(3.5)_u \qquad x_u(t) = H(t) - \int_0^t R_u(t, s) \, H(s) \, ds, \qquad t \geq 0.$$

Now we are ready for the following

COROLLARY 3.2. *Assume that* $K(t, s, u)$ *is defined and continuous on* $E_\infty \times R^n$ *and such that for each* $\varepsilon > 0$ *there exists a function* $\varphi_\varepsilon : R_+ \to R_+$, *bouded and integrable over* R_+, *and such that*

$$(3.6) \qquad \|K(t, \tau, u)\| \leq \varphi_\varepsilon(\tau) \quad , \quad (t, \tau, u) \in E_\infty \times R^n \quad \textit{with} \quad u \in S_\varepsilon .$$

Then the zero solution of the system $(^*)$ (*corresponding to* $H(t) \equiv 0$) *is stable in the sense of Definition 2.1, and with respect to the space* LC_b.

Proof. Assume that $u \in C_b[R_+, R^n]$ and that $\|u\|_{R_+} \leq \varepsilon$. Then $\|K(t, \tau, u(\tau))\| \leq \varphi_\varepsilon(\tau)$ for any $(t, \tau) \in E_\infty$. Thus, from $(3.4)_u$ we obtain

$$(3.7) \qquad \|R_u(t, s)\| \leq \|K(t, s, u(s))\| + \int_s^t \|K(t, \tau, u(\tau))\| \, \|R(\tau, s)\| \, d\tau$$

$$\leq \varphi_\varepsilon(s) + \int_s^t \varphi_\varepsilon(\tau) \, \|R(\tau, s)\| \, d\tau,$$

from which, applying Gronwall's inequality, keeping $s \leq t$ fixed, we get

$$(3.8) \qquad \|R_u(t, s)\| \leq \varphi_\varepsilon(s) \exp \left\{ \int_s^t \varphi_\varepsilon(\tau) \, d\tau \right\} \leq M,$$

where M is a nonnegative constant. Now let $H \in LC_b$ be such that $\| H \|_1 \leq \delta(\varepsilon) = \varepsilon/(1 + M)$. Then from $(3.5)_u$ we have

$$(3.9) \qquad \| x_u \|_{R_+} \leq \| H \|_{R_+} + M \int_0^\infty \| H(\tau) \| \, d\tau \leq (1 + M) \| H \|_1 \leq \varepsilon .$$

It follows that the systems $(^*)_u$ are iso-stable, and, by Theorem 3.1 (modified for the space LC_b), that the system $(^*)$ is stable with respect to LC_b.

The following example illustrates the above corollary. Consider the system $(^*)$ with

$$H(t) = \begin{bmatrix} 1/(t^2 + 1) \\ e^{-t} \end{bmatrix} , \qquad K(t, s, u) = \begin{bmatrix} e^{-(2t-s)} u_1 & u_2/(s^2 + t^2 + 1) \\ \sin(u_1 + t u_2) e^{-s^2} & 0 \end{bmatrix} .$$

Here we have $\| K(t, \tau, u) \| \leq \varphi_\varepsilon(\tau)$ for every $(t, \tau, u) \in E_\infty \times R^n$ with $\| u \| = \| (u_1, u_2) \| \leq \varepsilon$, where $\varphi_\varepsilon(\tau) = \max \{ [e^{-\tau} + 1/(\tau^2 + 1)] \varepsilon, e^{-\tau^2} \}$.

In the following corollary the integral operator in $(^*)_u$ is a contraction mapping on C_b.

COROLLARY 3.2. *Assume that the* $n \times n$ *matrix* $K(t, s, u)$ *is defined and continuous on* $E_\infty \times R^n$, *and such that for each* $\varepsilon > 0$,

$$\sup_{\substack{t \in R_+ \\ u \in S}} \int_0^t \| K(t, s, u(s)) \| \, ds \leq \varphi(\varepsilon) ,$$

where $\varphi(\varepsilon) : R_+ \setminus \{0\} \to (0, 1)$ *depends only on* ε.

Then the zero solution of the system $(^*)$ (*corresponding to* $H(t) \equiv 0$) *is stable in the sense of Definition 2.1.*

Proof. Given $\varepsilon > 0$ and $u \in C_b$ with $\| u \|_{R_+} \leq \varepsilon$, the operator

$$(3.10) \qquad (T_u f)(t) = \int_0^t K(t, s, u(s)) f(s) \, ds$$

is a contraction on C_b. Namely.

$$(3.11) \qquad \| T_u f_1 - T_u f_2 \|_{R_+} \leq \varphi(\varepsilon) \| f_1 - f_2 \|_{R_+} .$$

It follows from the contraction mapping principle that the system $(^*)_u$ has a solution $x_u \in C_b$ for every $H \in C_b$ and every $u \in C_b$. Moreover, this solution satisfies

$$(3.12) \qquad \| x_u \|_{R_+} \leq \| H \|_{R_+} + \varphi(\varepsilon) \| x_u \|_{R_+} ,$$

or

$$(3.13) \qquad \| x_u \|_{R_+} \leq [1/(1 - \varphi(\varepsilon))] \| H \|_{R_+} ,$$

from which follows immediately the iso-stability of the systems $(^*)_u$.

It is obvious that the method developed here can be applied to other kinds of stability of systems of the form (*). We omit these extensions since the corresponding proofs are identical to that of Theorem 3.1.

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