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# Rendiconti

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## Remarks on some boundedness theorems of Ezeilo and Tejumola

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**Equazioni differenziali.** — *Remarks on some boundedness theorems of Ezeilo and Tejumola.* Nota di H. O. TEJUMOLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore estende un risultato di definitiva limitatezza delle soluzioni di un'equazione differenziale del quarto ordine non lineare da lui precedentemente ottenuto in collaborazione con J. O. C. Ezeilo.

In the paper [2] we considered the differential equation

$$x^{IV} + a_1 \dddot{x} + a_2 \ddot{x} + \varphi(x)\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1}$$

where $a_1$, $a_2$, $a_4$ are constants and the functions $\varphi$ and $p$ are continuous. It was shown that all solutions of (1) are ultimately bounded if $a_1 > 0$, $a_2 > 0$, $a_4 > 0$ and if

$$\varphi(x) > 0 \quad , \quad a_1 a_2 \varphi(x) - \varphi^2(x) - a_1^2 a_4 > \delta \qquad \text{for} \quad |x| \geq 1, \tag{2}$$

$$|p(t, x, y, z, u)| \leq A_0 \qquad \text{for all} \quad t, x, y, z, u, \tag{3}$$

where $\delta > 0$, $A_0 \geq 0$ are constants.

The object of this note is to point out that this result extends readily to an equation (1) with the constant $a_2$ replaced by a function of $\dot{x}$ which is not necessarily nearly constant. A similar consideration applies to the equations studied in [3].

Consider the equation

$$x^{IV} + a_1 \dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{4}$$

in which $\psi$ is a continuous function, $a_1$, $a_4$ are, as before, constants and $\varphi$, $p$ are continuous. The following result holds.

THEOREM 1. *Let*

$$\Psi(y) = \int_0^y \psi(s)\, ds \tag{5}$$

*and suppose there is a constant* $a_2 > 0$ *such that*

$$\{\Psi(y) - a_2 y\} = o(1) \qquad as \quad |y| \to \infty. \tag{6}$$

*Let* $\varphi$, $p$ *satisfy conditions* (2) *and* (3). *Then there is a constant* $D > 0$ *whose magnitude depends only on* $a_1$, $a_2$, $a_4$, $\delta$, $A_0$ *and* $\Psi$ *such that every solution*

(*) Nella seduta del 9 febbraio 1974.

$x(t)$ *of* (4) *ultimately satisfies*

(7) $|x(t)| \leq D \quad , \quad |\dot{x}(t)| \leq D \quad , \quad |\ddot{x}(t)| \leq D \quad , \quad |\dddot{x}(t)| \leq D.$

Ezeilo's example [1; § 1] of a function $\psi(y)$ defined by

$$\psi(y) = a_2 + \frac{1}{2}\pi e^y \sin\left(\frac{1}{2}\pi e^y\right)$$

and for which condition (6) holds shows that (6) does not necessarily imply that $\Psi(y)$ is nearly constant or ultimately positive. A consequence of this condition is the estimate

(8) $$|\Psi(y) - a_2 y| \leq M \qquad \text{for all } y,$$

for some finite constant M.

The proof of Theorem 1 is essentially the same as that of [2] except for some modifications which we now point out. Take (4) in the system form

$$\dot{x} = y \quad , \quad \dot{y} = z \quad , \quad \dot{z} = u - a_1 z - \Phi(x) - \{\Psi(y) - a_2 y\},$$

(9) $$\dot{u} = -a_2 z - a_4 x + p(t, x, y, z, v),$$

$$\Phi(x) = \int_0^x \varphi(s)\, ds \quad , \quad v = u - a_1 z - \Phi(x) - \{\Psi(y) - a_2 y\}$$

and use the function $V = V_1 + V_2 + V_3$ defined by (4.3)-(4.6) of [2] but with $a_1, a_2, a_4$ playing the roles of $a, b$ and $c$ respectively. For precisely the same reasons in [2, §5] V satisfies

(10) $$V(x, y, z, u) \to +\infty \qquad \text{as} \quad x^2 + y^2 + z^2 + u^2 \to \infty.$$

Because of the term $-\{\Psi(y) - a_2 y\}$ in (9) above, which is absent in (4.1) of [2], the expression (6.2) of [2] for $\dot{U}_1$ will have to be augmented by

$$-\{a_2 x + a_1 y + z\}[\Psi(y) - a_2 y],$$

so that, in view of (8), the estimate (6.6) of [2] would now read

(11) $$\dot{U}_1 \leq -D_8(x^2 + z^2) + D_9(|x| + |y| + |z| + |u| + 1),$$

for some constant $D_9 > 0$ which depends also on M. The other relevant details which lead to the proof of (4.8) of [2]:

(12) $$\dot{V}^* \leq -1 \qquad \text{provided} \quad x^2 + y^2 + z^2 + u^2 \geq D_6$$

are as in [2; §7]; the term $D_9|z|$ in (11) being compensated for by $-D_8 z^2$.

Corresponding to the equation (1.4) of [3] we consider

(13) $$x^{(IV)} + a_1 \dddot{x} + \psi(\dot{x})\ddot{x} + g(\dot{x}) + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

where $a_1, a_4$ are constants and the functions $\psi$, $g$ and $p$ are continuous.

THEOREM 2. *Let* $\Psi$ *be defined by* (5) *and suppose there is a constant* $a_2 > 0$ *such that conditions* (6) *holds. Suppose further that* $a_1 > 0$, $a_4 > 0$ *and that*

(i) *there are constants* $\eta_0 > 0$, $d_1 > 0$ *such that*

$$g(y)/y > 0 \qquad (|y| \geq \eta_0),$$

$$a_1 a_2 \frac{g(y)}{y} - \left(\frac{g(y)}{y}\right)^2 - a_1^2 a_4 \geq d_1 \qquad (|y| \geq \eta_0),$$

(ii) $p(t, x, y, z, u)$ *satisfies* (3). *Then there is a constant* $D > 0$ *depending only on* $a_1, a_2, a_4, \eta_0, d_1, A_0, \Psi$ *and* $g$ *such that every solution* $x(t)$ *of* (1.3) *ultimately satisfies* (7).

In the special case $\psi(\dot{x}) \equiv a_2$, Theorem 2 reduces to [3, Theorem 1].

Let $V_0 = V_0(x, y, z, u)$ be the function (4.1) of [3] and consider (13) in the system form

$$\dot{x} = y \quad , \quad \dot{y} = z \quad , \quad \dot{z} = u - \{\Psi(y) - a_2 y\},$$
$$\dot{u} = -a_1 u - a_2 z - \{g(y) - a_1(\Psi(y) - a_2 y)\} - a_4 x + p(t, x, y, z, v),$$
$$v = u - \{\Psi(y) - a_2 y\}.$$

Then, in this case, the expression (4.5) of [3] for $\dot{V}_0$ will have to be augmented by

$$-[2 a_4 x + a_2 z - a_1 u]\{\Psi(y) - a_2 y\},$$

so that, by (8), the estimate (4.3) of [3] now reads

$$\dot{V}_0 \leq -D_3(y^2 + u^2) + D_4(|x| + |y| + |z| + |u| + 1) \tag{14}$$

for an appropriate choice of $D_4 > 0$ which also depends on M. In order to take care of the term $D_4 |x|$ in (14) (but which is absent in (4.3) of [3]), redefine the functions $V_1 = V_1(x, u)$, $V_2 = V_2(y, z)$ given by (5.2) and (5.3) of [3] as follows.

$$V_1 = \begin{pmatrix} \lambda u \operatorname{sgn} x, & |x| \geq |u| \\ \lambda x \operatorname{sgn} u, & |u| \geq |x| \end{pmatrix}, \qquad \lambda = a_4^{-1} D_4 + 1,$$

$$V_2 = \begin{pmatrix} -(2 D_4 + \lambda a_2) y \operatorname{sgn} z, & |z| \geq |y| \\ -(2 D_4 + \lambda a_2) z \operatorname{sgn} y, & |y| \geq |z| \end{pmatrix}.$$

Then, as in [3, §5], one shows that $V = V_0 + V_1 + V_2$ satisfies (10) and (12) for a suitable choice of the constant $D_6$.

Our last result concerns the equation

$$x^{IV} + f(\ddot{x})\dddot{x} + \psi(\dot{x})\ddot{x} + a_3 \dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{15}$$

in which $a_3$, $a_4$ are constants and $f$, $\psi$ and $p$ are continuous functions

THEOREM 3. *Let $a_2 > 0$ be a constant such that the function $\Psi$ defined by* (5) *satisfies* (6) *and suppose that*

(i) *there are constants $\xi_0 > 0$, $d_2 > 0$ such that*

$$f(z) > 0 \qquad (|z| \geq \xi_0),$$
$$a_2 a_3 f(z) - a_3^2 - a_4 f^2(z) \geq d_2 \quad (|z| \geq \xi_0),$$

(ii) *$p(t, x, y, z, u)$ satisfies* (3). *Then there is a constant $D > 0$ whose magnitude depends only on $a_2, a_3, a_4, \xi_0, d_2, A_0, f$ and $\Psi$ such that every solution $x(t)$ of* (15) *ultimately satisfies* (7).

Take (15) in the system form

$$\dot{x} = y \quad , \quad \dot{y} = z \quad , \quad \dot{z} = u - \{\Psi(y) - a_2 y\},$$
$$\dot{u} = -f(z)[u - \{\Psi(y) - a_2 y\}] - a_2 z - a_3 y + a_4 x + p(t, x, y, z, v),$$
$$v = u - \{\Psi(y) - a_2 y\},$$

and consider the function $V = U_0 + U_1 + U_2$ given by (10.1) of [3] but with $f$ playing the role of $\psi$. Here the expression (9.2) of [3] for $\dot{U}_0$ has to be augmented by

$$-[a_2 a_4 x + a_2 a_3 y + (a_2^2 - 2a_4) z + a_3 u - f(z)\{2a_4 y + a_2 u\}]\{\Psi(y) - a_2 y\}$$

and the estimate (9.6) of [3]:

$$\dot{U}_0 \leq -D_4(y^2 + z^2) + D_5(|x| + |y| + |z| + |u| + 1)$$

still holds for $\dot{U}_0$ (for a suitable choice of $D_5 > 0$) since $|f(z)|$ is bounded and $\Psi$ satisfies (8). The remainder of the arguments employed in § 10 and 11 of [3] to show that $V = U_0 + U_1 + U_2$ satisfies (10) and (12) carry over to the present case.

*Remark.* Theorem 3 extends to an equation

$$x^{IV} + f(\dot{x}, \ddot{x})\dddot{x} + \psi(\dot{x})\ddot{x} + a_3\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

in which $f$ depends on $\ddot{x}$ as well as on $\dot{x}$ provided the corresponding restrictions on $\psi(\dot{x}, \ddot{x})$ in [3, Theorem 3] are placed on $f(\dot{x}, \ddot{x})$ and $\Psi$, $p$ satisfies (8) and (3) respectively.

## REFERENCES


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