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Some Results for Surfaces with Flat Normal Connection

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Geometria differenziale. — *Some Results for Surfaces with Flat Normal Connection* (*). Nota di BANG-YEN CHEN, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Vengono classificate le superficie di uno spazio euclideo m-dimensionale dotate di connessione normale piatta, con lo studio di opportune equazioni alle derivate parziali. Alcuni casi particolari vengono approfonditi, facendo varie applicazioni.

§ 1. INTRODUCTION

Let x: $M \to R^m$ be an isometrical immersion of a surface M in an m-dimensional Riemannian manifold, R^m, and let ∇ and ∇' be the covariant differentiations of M and R^m respectively. Let X and Y be two tangent vector fields on M. Then the second fundamental form h is given by

$$\nabla'_X Y = \nabla_X Y + h(X, Y). \tag{1.1}$$

It is well-known that $h(X, Y)$ is a normal vector field on M and is symmetric on X and Y. Let ξ be a normal vector field on M, we write

$$\nabla'_X \xi = -A_\xi(X) + D_X \xi, \tag{1.2}$$

where $-A_\xi(X)$ and $D_X \xi$ denote the tangential and normal components of $\nabla'_X \xi$. Then we have

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle, \tag{1.3}$$

where $\langle\, ,\rangle$ denotes the scalar product in R^m. A normal vector field ξ on M is said to be *parallel* (in the normal bundle) if $D\xi = 0$. The *mean curvature vector* H is defined by

$$H = (\text{trace } h)/2. \tag{1.4}$$

The length of H is called the *mean curvature* of M. A surface M in R^m is said to be minimal if $H = 0$ identically. If the mean curvature vector H is nowhere zero and the second fundamental form h satisfies

$$\langle h(X, Y), H\rangle = \lambda \langle X, Y\rangle \tag{1.5}$$

for all tangent vectors X, Y on M, then M is said to be *pseudo-umbilical*.

Let R', R and R^N be the curvature tensors associated with connections ∇', ∇ and D respectively. For example, R^N is given by

$$R^N(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

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For a surface M in R^m, if the curvature tensor R^N vanishes identically, then M is said to have *flat normal connection*, and if the curvature tensor R vanishes identically, then M is said to be *flat*. It is known that: the only minimal surfaces in a euclidean space E^m with flat normal connection are surfaces in a 3-dimensional linear subspace of E^m; the only flat surfaces in E^4 with parallel mean curvature vector are open pieces of product surface of two plane circles (including straight lines); for surfaces in E^m, the flatness of the normal connection is equivalent to the absolute parallelism of the normal bundle and it is also equivalent to the simultaneously diagonalizability of the second fundamental tensors. From these facts, it seems to be interesting to study surfaces in a euclidean space with flat normal connection.

§ 2. FUNDAMENTAL THEOREM FOR SURFACES WITH FLAT NORMAL CONNECTION

Let M be a surface in an m-dimensional Riemannian manifold R^m, X , Y , W , Z any tangent vector fields on M and let ξ and η be any normal vector fields of M in R^m. Then the Gauss and Ricci equations are given by

$$\begin{aligned}(2.1)\qquad \langle R'(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle &+ \langle h(X, Z), h(Y, W)\rangle \\ &- \langle h(Y, Z), h(X, W)\rangle ,\end{aligned}$$

$$\begin{aligned}(2.2)\qquad \langle R'(X, Y)\xi, \eta\rangle = \langle R^N(X, Y)\xi, \eta\rangle &+ \langle h(A_\xi(X), Y), \eta\rangle \\ &- \langle h(A_\xi(Y), X), \eta\rangle\end{aligned}$$

respectively. For the second fundamental form h, we define the covariant derivative, denoted by $\overline{\nabla}_X h$, to be

$$(2.3)\qquad (\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

If the ambient space R^m is euclidean, then the Codazzi equation is given by

$$(2.4)\qquad (\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z).$$

The Gauss and Ricci equations reduce to

$$(2.5)\qquad \langle R(X, Y)Z, W\rangle = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle ,$$

$$(2.6)\qquad \langle R^N(X, Y)\xi, \eta\rangle = \langle h(A_\xi(Y), X), \eta\rangle - \langle h(A_\xi(X), Y), \eta\rangle .$$

If the normal connection is flat, $R^N = 0$, then, by Propositions 1.1 of Chapter 4 in [1], we see that there exist (at least locally) $m - 2$ mutually orthogonal unit normal vector fields $\xi_1, \cdots, \xi_{m-2}$ such that

$$(2.7)\qquad D\xi_1 = D\xi_2 = \cdots = D\xi_{m-2} = 0 .$$

Moreover, from (2.6) and the fact $R^N = 0$, we see that

$$(2.8)\qquad [A_\alpha, A_\beta] = 0 , \qquad \alpha, \beta = 1, \cdots, m-2 ,$$

where $A_\alpha = A_{\xi_\alpha}$. From (2.8) we see that there exist two mutually orthogonal unit tangent vector fields e_1, e_2 on M such that, with respect to frame $e_1, e_2, \xi_1, \cdots, \xi_{m-2}$, the second fundamental tensors A_α are given by

$$A_\alpha = \begin{pmatrix} a_\alpha & 0 \\ 0 & b_\alpha \end{pmatrix}, \qquad \alpha = 1, 2, \cdots, m-2. \tag{2.9}$$

Since M is two dimensional and our study is local, we may assume that M is covered by isothermal coordinates (x, y) such that the metric on M has the form $ds^2 = E(dx^2 + dy^2)$. In the following, we denote the coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$ by X_1 and X_2 respectively, and we put

$$L = h(X_1, X_1) \quad , \quad M = h(X_1, X_2) \quad , \quad N = h(X_2, X_2) \tag{2.10}$$

and

$$\nabla_{X_j} X_i = \Sigma \Gamma_{ji}^k X_k . \tag{2.11}$$

Then we have

$$\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{X_1 E}{2E} \quad , \quad \Gamma_{22}^2 = \Gamma_{12}^1 = -\Gamma_{11}^2 = \frac{X_2 E}{2E} . \tag{2.12}$$

Therefore the Codazzi equation (2.4) gives

$$D_{X_2} L - D_{X_1} M = (X_2 E) H , \tag{2.13}$$

$$D_{X_2} M - D_{X_1} N = -(X_1 E) H . \tag{2.14}$$

Since X_1 and X_2 are orthonormal, we may put

$$X_1 = \cos\theta e_1 + \sin\theta e_2 \quad , \quad X_2 = -\sin\theta e_1 + \cos\theta e_2 . \tag{2.15}$$

Then, with respect to the frame $X_1, X_2, \xi_1, \cdots, \xi_{n-2}$, the second fundamental tensors are given by

$$A_\alpha = \begin{pmatrix} a_\alpha \cos^2\theta + b_\alpha \sin^2\theta & \frac{1}{2}(b_\alpha - a_\alpha)\sin 2\theta \\ \frac{1}{2}(b_\alpha - a_\alpha)\sin 2\theta & a_\alpha \sin^2\theta + b_\alpha \cos^2\theta \end{pmatrix} . \tag{2.16}$$

From (2.7), (2.16), the Codazzi equations (2.13) and (2.14) reduce to

$$2\frac{\partial}{\partial y}(a_\alpha \cos^2\theta + b_\alpha \sin^2\theta) - \frac{\partial}{\partial x}\{(b_\alpha - a_\alpha)\sin 2\theta\} = \frac{\partial E}{\partial y}(a_\alpha + b_\alpha) , \tag{2.17}$$

$$2\frac{\partial}{\partial x}(a_\alpha \sin^2\theta + b_\alpha \cos^2\theta) - \frac{\partial}{\partial y}\{(b_\alpha - a_\alpha)\sin 2\theta\} = \frac{\partial E}{\partial x}(a_\alpha + b_\alpha) . \tag{2.18}$$

Moreover, since $ds^2 = E(dx^2 + dy^2)$, by virtue of (2.9) Gauss equation (2.5) reduces to

$$\sum_{\alpha=1}^{m-2} a_\alpha b_\alpha = -\frac{1}{2E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln E. \tag{2.19}$$

Consequently, for a surface M in E^m with isothermal coordinates (x, y) and metric $ds^2 = E(ds^2 + dy^2)$, if the normal connection is flat, then locally there exist $2m-3$ functions $(\theta, a_\alpha, b_\alpha)$ satisfying the system (2.17), (2.18) and (2.19).

Conversely, suppose that D is a disc around the origin of the (x, y)-plane with isothermal metric $ds_D^2 = E(dx^2 + dy^2)$. Suppose that there exist $2m-3$ functions $(\theta, a_\alpha, b_\alpha)$, $\alpha = 1, \cdots, m-2$, satisfying (2.17), (2.18) and (2.19).

Let $N = D \times E^{m-2}$ be the Riemannian product of D and euclidean $(m-2)$-space E^{m-2}. We consider N as the total space of a vector bundle over D with metric ds_D^2. Then N is a Riemannian vector bundle with a usual connection D. Let $\xi_1, \cdots, \xi_{m-2}$ be an orthonormal parallel frame of N. Such a parallel frame exists because D is the usual flat connection on E^{m-2}. We define $X_1 = \partial/\partial x$, $X_2 = \partial/\partial y$ and

$$
\begin{aligned}
h(X_1, X_1) &= \sum_\alpha (a_\alpha \cos^2\theta + b_\alpha \sin^2\theta)\,\xi_\alpha, \\
h(X_1, X_2) &= \frac{1}{2}\sum_\alpha \{(b_\alpha - a_\alpha)\sin 2\theta\}\,\xi_\alpha \\
h(X_2, X_2) &= \sum_\alpha (a_\alpha \sin^2\theta + b_\alpha \cos^2\theta)\,\xi_\alpha.
\end{aligned}
\tag{2.20}
$$

Then, by the fundamental theorem of submanifolds (see, for instance, p. 48 of [1]), we see that there exists an isometric immersion of (D, ds_D^2) into E^m, with N as its normal bundle and h as its second fundamental form, if and only if E and h satisfy the Gauss, Ricci and Codazzi equations (2.1), (2.2) and (2.4). Since $R' = 0$, Gauss equation follows immediately from (2.19); Codazzi equation reduces to (2.17) and (2.18); Ricci equation (2.2) is

$$
-R^N(X_i, X_j)\,\xi_\alpha = h(A_\alpha(X_i), X_j) - h(A_\alpha(X_j), X_i),
$$

for $i, j = 1, 2$ and $\alpha = 1, 2, \cdots, m-2$. The left-hand side is zero because ξ_α are parallel and the right-hand side vanishes on account of the definition (2.20) of h.

To sum up, we have the following fundamental theorem for surfaces with flat normal connection:

THEOREM 1. *Let* M *be a surface isometrically immersed in a euclidean m-space* E^m *with locally isothermal metric* $ds^2 = E(dx^2 + dy^2)$. *If the normal connection of* M *in* E^m *is flat, then, locally, there exist* $2m-3$ *functions* $(\theta(x, y), a_\alpha(x, y), b_\alpha(x, y))$, $\alpha = 1, 2, \cdots, m-2$, *satisfying* (2.17), (2.18) *and* (2.19).

Conversely, let D *be a unit disc in the* (x, y)*-plane with an isothermal metric* $ds^2 = E(dx^2 + dy^2)$. *If there exist* $2m-3$ *functions* $(\theta, a_\alpha, b_\alpha)$ *on* D *satisfying* (2.17), (2.18) *and* (2.19), *then there exists an isometric immersion of* (D, E) *into* E^m *with flat normal connection and with second fundamental form h given by* (2.20), *where* ξ_α *are parallel orthonormal normal vector fields. Such isometric immersion is unique up to motions in* E^m.

The first part of this theorem has already been proved and the second part follows immediately from the rigidity theorem of submanifolds (see, for instance, p. 49 of [1]).

§ 3. Flat surfaces in E^m with flat normal connection

In Theorem 1, we have proved that, locally, surfaces in a Euclidean m-space E^m with flat normal connection are in one-to-one correspondence with the solutions $(\theta , a_\alpha , b_\alpha)$ of system (2.17), (2.18) and (2.19). In this Section, we shall try to obtain such solutions for flat surfaces in E^4 with certain properties.

Now, we consider a flat pseudo-umbilical surface M in E^4, with mean curvature vector H. Then it is clear that the normal connection is flat. Let ξ_1 , ξ_2 be two mutually orthogonal unit normal vector fields such that

$$H = | H | \xi_1 , \tag{3.1}$$

and let e_1 , e_2 be two mutually orthogonal unit tangent vector fields which are in the principal directions of A_2. Then, with respect to the frame field $e_1 , e_2 , \xi_1 , \xi_2$, we have

$$A_3 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad , \quad A_4 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad , \quad a = | H | . \tag{3.2}$$

Since the normal connection is flat, there exist uniquely two mutually orthogonal unit normal vector fields ξ_1 , ξ_2 such that $\bar{\xi}_1 , \bar{\xi}_2$ are parallel. Let $\varphi = \varphi (x , y)$ be a function defined by

$$\bar{\xi}_1 = \cos \varphi \, \xi_1 + \sin \varphi \, \xi_2 \quad , \quad \bar{\xi}_2 = - \sin \varphi \, \xi_1 + \cos \varphi \, \xi_2 , \tag{3.3}$$

with $\varphi = 0$ at a fixed point P in M. Now we define a function $\theta = \theta (x , y)$ by

$$X_1 = \frac{\partial}{\partial x} = \cos \theta \, e_1 + \sin \theta \, e_2 \quad , \quad X_2 = \frac{\partial}{\partial y} = - \sin \theta \, e_1 + \cos \theta \, e_2 . \tag{3.4}$$

Without loss of generality, we may assume that $(x , y) = (0 , 0)$ at P and $\theta (0 , 0) = 0$. Then, with respect to the frame field $X_1 , X_2 , \bar{\xi}_1 , \bar{\xi}_2$, the second fundamental tensors are given by

$$A_{\bar{1}} = \begin{pmatrix} a \cos \varphi + a \sin \varphi \cos 2\theta & - a \sin \varphi \sin 2\theta \\ - a \sin \varphi \sin 2\theta & a \cos \varphi - a \sin \varphi \cos 2\theta \end{pmatrix} , \tag{3.5}$$

$$A_{\bar{2}} = \begin{pmatrix} - a \sin \varphi + a \cos \varphi \cos 2\theta & - a \cos \varphi \sin 2\theta \\ - a \cos \varphi \sin 2\theta & - a \sin \varphi - a \cos \varphi \cos 2\theta \end{pmatrix} . \tag{3.6}$$

Since $\bar{\xi}_1$, $\bar{\xi}_2$ are parallel and $E=1$, Codazzi equations (2.13) and (2.14) reduce to

$$\begin{aligned}(3.8)\quad &\frac{\partial a}{\partial y}(\cos\varphi+\sin\varphi\cos 2\theta)+\frac{\partial\varphi}{\partial y}(-a\sin\varphi+a\cos\varphi\cos 2\theta)-\frac{\partial\theta}{\partial y}(2a\sin\varphi\sin 2\theta)\\ &=-\frac{\partial a}{\partial x}(\sin\varphi\sin 2\theta)-2\frac{\partial\theta}{\partial x}(a\sin\varphi\cos 2\theta)-\frac{\partial\varphi}{\partial x}(a\cos\varphi\sin 2\theta),\end{aligned}$$

$$\begin{aligned}(3.9)\quad &-\frac{\partial a}{\partial y}(\sin\varphi\sin 2\theta)-2\frac{\partial\theta}{\partial y}(a\sin\varphi\cos 2\theta)-\frac{\partial\varphi}{\partial y}(a\cos\varphi\sin 2\theta)\\ &=\frac{\partial a}{\partial x}(\cos\varphi-\sin\varphi\cos 2\theta)-\frac{\partial\varphi}{\partial x}(a\sin\varphi+a\cos\varphi\cos 2\theta)+2\frac{\partial\theta}{\partial x}(a\sin\varphi\sin 2\theta),\end{aligned}$$

$$\begin{aligned}(3.10)\quad &-\frac{\partial a}{\partial y}(\sin\varphi+\cos\varphi\cos 2\theta)-\frac{\partial\varphi}{\partial y}(a\cos\varphi+a\sin\varphi\cos 2\theta)-2\frac{\partial\theta}{\partial y}(a\cos\varphi\sin 2\theta)\\ &=-\frac{\partial a}{\partial x}(\cos\varphi\sin 2\theta)+\frac{\partial\varphi}{\partial x}(a\sin\varphi\sin 2\theta)-2\frac{\partial\theta}{\partial x}(a\cos\varphi\cos 2\theta),\end{aligned}$$

$$\begin{aligned}(3.11)\quad &-\frac{\partial a}{\partial y}(\cos\varphi\sin 2\theta)-2\frac{\partial\theta}{\partial y}(a\cos\varphi\cos 2\theta)+\frac{\partial\varphi}{\partial y}(a\sin\varphi\sin 2\theta)\\ &=-\frac{\partial a}{\partial x}(\sin\varphi+\cos\varphi\cos 2\theta)+2\frac{\partial\theta}{\partial x}(a\cos\varphi\sin 2\theta)-\frac{\partial\varphi}{\partial x}(a\cos\varphi-a\sin\varphi\cos 2\theta).\end{aligned}$$

By computing $\cos\varphi\times(3.8)-\sin\varphi\times(3.10)$, we find

$$(3.12)\qquad \frac{\partial\ln a}{\partial y}=-\sin 2\theta\frac{\partial\varphi}{\partial x}-\cos 2\theta\frac{\partial\varphi}{\partial y}\,.$$

By computing $\cos\varphi\times(3.9)+\sin\varphi\times(3.11)$, we find

$$(3.13)\qquad \frac{\partial\ln a}{\partial x}=\cos 2\theta\frac{\partial\varphi}{\partial x}-\sin 2\theta\frac{\partial\varphi}{\partial y}\,.$$

By computing $\sin\varphi\times(3.8)+\cos\varphi\times(3.10)$ and applying (3.12) and (3.13), we find

$$(3.14)\qquad \frac{\partial\varphi}{\partial y}=\cos 2\theta\frac{\partial\theta}{\partial x}-\sin 2\theta\frac{\partial\theta}{\partial y}\,.$$

Similarly, by computing $\sin\varphi\times(3.9)-\cos\varphi\times(3.11)$ and applying (3.12) and (3.13), we find

$$(3.15)\qquad \frac{\partial\varphi}{\partial x}=\sin 2\theta\frac{\partial\theta}{\partial x}+\cos 2\theta\frac{\partial\theta}{\partial y}\,.$$

Now, by substituting (3.14) and (3.15) into (3.12) and (3.13), we find

$$(3.16)\qquad \frac{\partial\ln a}{\partial y}=-\frac{\partial\theta}{\partial x}\quad,\quad\frac{\partial\ln a}{\partial x}=\frac{\partial\theta}{\partial y}\,.$$

From (3.16) we have

$$\frac{\partial^2 \ln a}{\partial x^2} + \frac{\partial^2 \ln a}{\partial y^2} = 0 . \tag{3.17}$$

Since $ds^2 = E(dx^2 + dy^2)$ on M, (3.17) implies that $\ln a$ is harmonic, i.e., $\Delta \ln a = 0$, where Δ denotes the Laplacian on surface M. Consequently, we have:

PROPOSITION 2. *Let* M *be a flat pseudo-umbilical surface in* E^4. *Then the mean curvature* $a = |H|$ *satisfies*

$$\Delta \ln a = 0 . \tag{3.18}$$

As a consequence of Proposition 2, we have

THEOREM 3. *Let* M *be a complete flat pseudo-umbilical surface in* E^4. *If the mean curvature is bounded below from a positive number, then* M *is a Clifford torus*, i.e., M *is the product surface of two plane circles with the same radius.*

Proof. Since M is complete and flat, M is parabolic in the sense that there exists no nonconstant negative subharmonic function on M (see, for instance, chapter 1 of [1]). Since $\ln a$ is harmonic and hence subharmonic, the hypothesis implies that a is constant. Hence this theorem follows immediately from Proposition 2 and the following

THEOREM 4 [2]. *The only flat pseudo-umbilical surfaces with constant mean curvature in* E^4 *are open pieces of a Clifford torus.*

COROLLARY 1. *The only compact flat pseudo-umbilical surfaces in* E^4 *are Clifford tori.*

This corollary follows immediately from Theorem 3.

§ 4. LOCAL CLASSIFICATION THEOREM FOR FLAT PSEUDO-UMBILICAL SURFACES

From Theorem 3 we see that the only compact flat pseudo-umbilical surfaces in E^4 are Clifford tori. In this section we shall give a classification theorem for the local case.

Let M be a flat pseudo-umbilical surface in E^4, with mean curvature $a = |H|$. Then, by § 3, we see that, locally, there exist two functions $\theta = \theta(x, y)$, $\varphi = \varphi(x, y)$ defined by (3.3) and (3.4), satisfying equations (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17) with the initial conditions

$$\theta(P) = 0 \quad , \quad \varphi(P) = 0 . \tag{4.1}$$

From (3.16) we find

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 , \tag{4.2}$$

where (x, y) are isothermal coordinates around P with $ds^2 = dx^2 + dy^2$. From (3.14) and (3.15) we find

$$\cos 2\theta \left[4 \frac{\partial\theta}{\partial x} \frac{\partial\theta}{\partial y} + \frac{\partial^2\theta}{\partial y^2} - \frac{\partial^2\theta}{\partial x^2}\right] = 2 \sin 2\theta \left[\left(\frac{\partial\theta}{\partial y}\right)^2 - \left(\frac{\partial\theta}{\partial x}\right)^2 - \frac{\partial^2\theta}{\partial x\,\partial y}\right]. \tag{4.3}$$

Consequently, we have:

PROPOSITION 5. *Let* M *be a flat pseudo-umbilical surface in* E^4 *and* P *be any point in* M. *Then there exist a neighbourhood* U *of* P, *isothermal coordinates* (x, y) *on* U *with metric* $ds^2 = dx^2 + dy^2$, *and a harmonic function* $\theta = \theta(x, y)$ *satisfying equation* (4.3).

Conversely, we shall prove the following existence theorem.

THEOREM 6. *Let* D *be the unit disc centered at the origin in the* (x, y)–*plane with the usual metric* $ds^2 = dx^2 + dy^2$. *Let* $\theta = \theta(x, y)$ *be a harmonic function on* D *satisfying equation* (4.3). *Then, up to motions and similarities in* E^4, *there exists one and only one isometric immersion of* (D, ds^2) *into* E^4 *such that*

(1) *the immersion is pseudo-umbilical, in particular, the normal connection is flat, and*

(2) θ *satisfies*

$$\frac{\partial}{\partial x} = \cos\theta e_1 + \sin\theta e_2 \quad , \quad \frac{\partial}{\partial y} = -\sin\theta e_1 + \cos\theta e_2 , \tag{4.4}$$

where e_1, e_2 *are orthonormal vector fields which are in the principal directions of the second fundamental tensors.*

Proof. Let D be the unit disc centered at the origin in the (x, y)–plane with the usual metric $ds^2 = dx^2 + dy^2$. Let $\theta = \theta(x, y)$ be a harmonic function which satisfies (4.3) on D. We consider a differential system given by

$$\frac{\partial\varphi}{\partial y} = \cos 2\theta \frac{\partial\theta}{\partial x} - \sin 2\theta \frac{\partial\theta}{\partial y} \quad , \quad \frac{\partial\varphi}{\partial x} = \sin 2\theta \frac{\partial\theta}{\partial x} + \cos 2\theta \frac{\partial\theta}{\partial y}. \tag{4.5}$$

Then (4.3) implies that the two equations of system (4.5) are coincident. Since D is simply-connected, there exists a unique function $\varphi = \varphi(x, y)$ on D with $\varphi(0, 0) = 0$. Now, since θ is harmonic, the following system is also coincident

$$\frac{\partial \ln a}{\partial y} = -\frac{\partial\theta}{\partial x} \quad , \quad \frac{\partial \ln a}{\partial x} = \frac{\partial\theta}{\partial y}. \tag{4.6}$$

Hence, up to a constant, there exists a unique function $a = a(x, y)$ satisfying (4.6). Now, let $a = a(x, y)$ be the given solution of (4.6). By using (4.5) and (4.6) we find

$$\frac{\partial \ln a}{\partial y} = -\sin 2\theta \frac{\partial\varphi}{\partial x} - \cos 2\theta \frac{\partial\varphi}{\partial y} , \tag{4.7}$$

$$\frac{\partial \ln a}{\partial x} = \cos 2\theta \frac{\partial\varphi}{\partial x} - \sin 2\theta \frac{\partial\varphi}{\partial y}. \tag{4.8}$$

Now, we define four functions a_1, a_2, b_1, b_2 by

$$a_1 = a(\cos\varphi + \sin\varphi) \quad , \quad a_2 = a(-\sin\varphi + \cos\varphi) ,$$
$$b_1 = a(\cos\varphi - \sin\varphi) \quad , \quad b_2 = a(-\sin\varphi - \cos\varphi) .$$

Then, by using (4.5), (4.7) and (4.8), we see that functions $(\theta, a_1, a_2, b_1, b_2)$ satisfy equations (2.17), (2.18) and (2.19) with $E = 1$. Hence, by Theorem 1, there exists an isometric immersion of (D, ds^2) into E^4 with second fundamental form h given by

$$\begin{aligned} h(X, X) &= \sum_{\alpha=1}^{2} (a_\alpha \cos^2\theta + b_\alpha \sin^2\theta)\, \xi_\alpha , \\ h(X, Y) &= \sum_{\alpha=1}^{2} (b_\alpha - a_\alpha) \sin 2\theta\, \xi_\alpha , \\ h(Y, Y) &= \sum_{\alpha=1}^{2} (a_\alpha \sin^2\theta + b_\alpha \cos^2\theta)\, \xi_\alpha , \end{aligned} \tag{4.9}$$

where ξ_1, ξ_2 are two orthonormal parallel normal vector fields. From (4.9), it is easy to see that the immersion is pseudo-umbilical and θ satisfies (4.4). This proves the first part of the theorem. The second part follows then immediately from the rigidity theorem of submanifolds.

From Proposition 5 and Theorem 6, we have

THEOREM 7. *Locally, flat pseudo-umbilical surfaces in* E^4 *are one-to-one correspondent to a pair* (θ, c), *where* θ *is a harmonic function satisfying* (4.3) *and* c *is a nonzero constant. Let the surface associated with* (θ, c) *be denoted by* $M(\theta, c)$. *Then* $M(\theta, c)$ *is an open piece of a Clifford torus if and only if* θ *is constant.*

Proof. The first part of this corollary follows from Proposition 5 and Theorem 6, where constant c comes from similarities. For the proof of the second part, if $M(\theta, c)$ is an open piece of a Clifford torus, the mean curvature vector is parallel. Hence φ is constant. Hence, from (4.5), θ is constant. Conversely, if θ is constant, then (4.6) implies that the mean curvature is constant. Thus, by Theorem 4, $M(\theta, c)$ is an open piece of a Clifford torus.

REMARK. It is very interesting to decide whether the only harmonic functions satisfying differential equation (4.3) are constant functions. If it is so, the local version of Corollary 1 holds. Otherwise, there exists some flat pseudo-umbilical surface in E^4 which is not an open piece of a Clifford torus.

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