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Generalized Picard Theorem for Ordinary Differential Equations

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Equazioni differenziali ordinarie. — *Generalized Picard Theorem for Ordinary Differential Equations.* Nota di TOMASO POMENTALE (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si prova un teorema di esistenza e unicità per il problema ai valori iniziali. Si fa uso di una funzione di iterazione « multipoint » soddisfacente a una condizione di Lipschitz generalizzata di tipo localmente monotono.

Let I and J be intervals defined respectively by $0 \leq x \leq a$, $\beta - b \leq \leq y \leq \beta + b$, $b > 0$, and let $F(x, y)$ be a continuous function on $I \times J$. We prove, in Theorem 2, the existence and uniqueness of a solution of the differential equation

$$y' = F(x, y) \tag{1}$$

with the condition

$$y(0) = \beta \tag{2}$$

using a generalization ((3) below) of the Picard method.

Let $f(x, z_1, \cdots z_p)$ be a function, continuous on $I \times J \times \cdots J$, satisfying

$$f(x, y, \cdots y) = F(x, y).$$

Given $p - 1$ initial functions $y_0(x), \cdots y_{p-2}(x)$ on I such that $y_i \in J$, $i = 0, 1 \cdots p - 2$ we formally define the iteration process

$$\begin{aligned} y'_{n+1} &= f(x, y_{n+1}, y_n \cdots y_{n-p+2}) \\ y_{n+1}(0) &= \beta. \end{aligned} \tag{3}$$

V. Capra [2] (1963) proved an existence and uniqueness theorem for (1), (2), using (3) with $p = 2$ and assuming that $f(x, z_1, z_2)$ is Lipschitz continuous with respect to z_1 and z_2. Recently a similar result was obtained, for $p = 2$, by O. Ashirov and Ya. D. Mamedov [1], who imposed on f a more general Lipschitz condition, precisely

$$|f(x, z_1, z_2) - f(x, \bar{z}_1, \bar{z}_2)| \leq \varphi(x, |z_1 - \bar{z}_1|, |z_2 - \bar{z}_2|)$$

where $\varphi(x, u_1, u_2)$ is continuous, nondecreasing in u_2, such that the equation

$$u'_1 = \varphi(x, u_1, u_2)$$

(*) CERN, Ginevra.
(**) Nella seduta del 20 aprile 1974.

has a solution $u_1(x)$, $0 \leq u_1 \leq 2b$, $0 \leq x \leq a$, satisfying $u_1(0) = 0$ for every continuous function $u_2(x)$, $0 \leq u_2 \leq 2b$, $0 \leq x \leq a$, and the equation

$$u' = \varphi(x, u, u)$$

has the unique solution $u(x) = 0$, $0 \leq x \leq a$ satisfying the condition $u(0) = 0$.

Our Theorem 2 considers the case $p \geq 2$ which gives multipoint iterations and allows, when f does not contain the variable z_1, the computation of the iterates y_n by quadratures as for the Picard method. Furthermore we impose on f a very general condition which includes the generalized Lipschitz condition of Ashirov and Mamedov. The following theorem is an extension of Theorem 1 given in [1]. It may easily be proved using the comparison theorems and the Ascoli Theorem given in [4]. It will be used in the proof of Theorem 2.

THEOREM 1. *Let f and F be defined as above and assume that f satisfies in $I \times J \times \cdots J$ the inequality*

$$|f(x, z_1, \cdots z_p) - f(x, \bar{z}_1, \cdots \bar{z}_p)| \leq \varphi(x, |z_1 - \bar{z}_1|, \cdots |z_p - \bar{z}_p|)$$

where $\varphi(x, u_1, \cdots u_p)$ is a continuous function on $0 \leq x \leq a$, $0 \leq u_i \leq 2b$, $i = 1, \cdots p$, nondecreasing in u_i, $i = 2, \cdots p$, such that the problem

$$u' = \varphi(x, u, \cdots u) \qquad u(0) = 0 \tag{4}$$

has the unique solution $u(x) = 0$, $0 \leq x \leq a$.

If the iterations defined by (3) remain in J for $0 \leq x \leq a$, there exists a unique solution $y(x)$, $0 \leq x \leq a$, $y \in J$ of (1) and (2) and the iterations converge uniformly to $y(x)$.

Remark. Let us define the following function $\bar{\varphi}(x, u_1, \cdots u_p)$ in the domain $0 \leq x \leq a$, $0 \leq u_i < \infty$, $i = 1, \cdots p$

$$\bar{\varphi}(x, u_1, \cdots u_p) = \varphi(x, u_1, \cdots u_p) \quad \text{when} \quad 0 \leq u_i \leq 2b \qquad i = 1, \cdots p$$

$$\bar{\varphi}(x, u_1, \cdots u_k, \cdots u_p) = \varphi(x, u_1, \cdots 2b, \cdots u_p) \quad \text{when} \quad u_k > 2b.$$

Then, since $\bar{\varphi}$ is bounded, there exists a constant $d \geq 2b$ such that $u^*(x) \leq d$, $0 \leq x \leq a$, where $u^* \geq 0$ is the maximum solution of

$$u_1' = \bar{\varphi}(x, u_1, d, \cdots d) \quad , \quad u_1(0) = 0.$$

We deduce that $0 \leq \varepsilon_{n+1}(x) \leq \varepsilon_n(x)$, $n = 0, 1 \cdots$, $0 \leq x \leq a$, where ε_{n+1} is the maximum solution of

$$\varepsilon_{n+1}' = \bar{\varphi}(x, \varepsilon_{n+1}, \varepsilon_n \cdots \varepsilon_{n-p+2})$$

$$\varepsilon_{n+1}(0) = 0$$

$$\varepsilon_0(x) = \varepsilon_1(x) \cdots = \varepsilon_{p-2}(x) = d, \qquad 0 \leq x \leq a.$$

The sequence $\{\varepsilon_n\}$ describes the rate of convergence of $\{y_n\}$, since it is easy to see that, if $y(x)$ is the solution of (1) and (2), then

$$|y_n(x) - y(x)| \leq \varepsilon_n(x), \qquad 0 \leq x \leq a.$$

We use the following definition: a function $\psi(x, u_1, \cdots u_p)$ is nondecreasing at $u_i = 0$, $i = 2, \cdots p$, if there exists an $\varepsilon > 0$ such that in the region $0 \leq x \leq a$, $0 \leq u_i \leq \varepsilon$, $i = 1, \cdots p$, ψ is nondecreasing in u_i, $i = 2, \cdots p$.

THEOREM 2. *Assume that* f *and* F *are defined as in Theorem* 1 *and, in* $I \times J \times \cdots J$,

$$|f(x, z_1, \cdots z_p) - f(x, \bar{z}_1, \cdots \bar{z}_p)| \leq \varphi(x, |z_1 - \bar{z}_1|, \cdots |z_p - \bar{z}_p|)$$

where φ *is a continuous function on* $I \times \bar{J} \times \cdots \bar{J}$, $\bar{J} = [0, 2b]$, *nondecreasing at* $u_i = 0$, $i = 2, \cdots p$, *and such that* (4) *has the unique solution* $u(x) = 0$. *Suppose that the iterations* (3) *remain in* J *for* $0 \leq x \leq a$, *then there exists a unique solution* $y(x)$, $0 \leq x \leq a$, $y \in J$ of (1) *and* (2) *and* $y_n(x) \to y(x)$ *uniformly.*

Proof. The theorem is a consequence of Theorem 1 if we set up a continuous function $\psi(x, u_1, \cdots u_p)$ nondecreasing in $u_2, \cdots u_p$, defined for $0 \leq x \leq a$, $0 \leq u_i \leq 2b$, $i = 1, \cdots p$ and such that

a) on the domain $I \times \bar{J} \times \cdots \bar{J}$

$$\varphi(x, u_1, \cdots u_p) \leq \psi(x, u_1, \cdots u_p)$$

b) the equation

$$u' = \psi(x, u, \cdots u)$$

with

$$u(0) = 0$$

has the unique solution $u(x) = 0$, $0 \leq x \leq a$.

Using φ, at first we set up a continuous function $\psi_1(x, u_1, \cdots u_p)$ nondecreasing in u_2 and satisfying *a*) and *b*). Using ψ_1 we set up $\psi_2(x, u_1, \cdots u_p)$, nondecreasing in u_2, u_3, satisfying *a*), *b*), and so on until $\psi = \psi_{p-1}(x, u_1, \cdots u_p)$.

Assume $(x, u_1, u_3, \cdots u_p)$ fixed and consider $\varphi(x, u_1, \cdots u_p)$ as function of u_2. If φ is nondecreasing in u_2 define

$$\psi_1(x, u_1, \cdots u_p) = \varphi(x, u_1, \cdots u_p), \qquad 0 \leq u_2 \leq 2b$$

otherwise let S be the set of the abscissas $u_2^1 < 2b$ of the local maxima $M(u_2^1)$ such that $M(u_2^1) \geq \varphi(x, u_1, \cdots u_p)$ for $u_2 \leq u_2^1$ (i.e. $M(u_2^1)$ is also absolute maximum in $[0, u_2^1]$). In the case where a local maximum corresponds to an interval, we define u_2^1 as the right end-point of the interval. For every $u_2^1 \in S$ define

$$\psi_1(x, u_1, \cdots u_p) = M(u_2^1)$$

for

$$u_2^1 \leq u_2 \leq u_2^2 \quad (u_2^2 \leq 2b)$$

where u_2^2 is either the point at which the straight line parallel to the u_2-axis and touching φ at M (u_2^1) crosses φ for $u_2 > u_2^1$, or $u_2^2 = 2b$ if such straight line does not cross φ for $u_2 > u_2^1$.

Define

$$\psi_1(x, u_1, \cdots u_p) = \varphi(x, u_1, \cdots u_p)$$

when u_2 does not belong to any of the above intervals.

ψ_1 which has domain $0 \leq x \leq a$, $0 \leq u_i \leq 2b$, $i = 1, \cdots p$, is nondecreasing in u_2 and continuous on $I \times J \times \cdots J$.

Condition *a*) for ψ_1 is satisfied, *b*) follows from the local monotonicity of φ, because $\psi_1(x, u, \cdots u) = \varphi(x, u, \cdots u)$ for $0 \leq x \leq a$, $0 \leq u \leq \varepsilon\,(< 2b)$. If we perform the same procedure taking ψ_1 instead of φ and consider the variable u_3 instead of u_2 we get a continuous function $\psi_2(x, u_1, \cdots u_p)$, nondecreasing in u_3. Moreover ψ_2 is also nondecreasing in u_2. In fact, if $\overset{*}{u_2} < \overset{**}{u_2}$, it is

$$\psi_1(x, u_1, \overset{*}{u_2}, u_3, \cdots u_p) \leq \psi_1(x, u_1, \overset{**}{u_2}, u_3, \cdots u_p)$$

then we have

$$\psi_2(x, u_1, \overset{**}{u_2}, u_3, \cdots u_p) \geq \psi_1(x, u_1, \overset{*}{u_2}, u_3, \cdots u_p)$$

but ψ_2 is nondecreasing in u_3, consequently we have also

$$\psi_2(x, u_1, \overset{**}{u_2}, u_3, \cdots u_p) \geq \psi_2(x, u_1, \overset{*}{u_2}, u_3, \cdots u_p).$$

In the same way we set up $\psi_3, \cdots \psi_{p-1}$ and define $\psi = \psi_{p-1}(x, u_1, \cdots u_p)$ which is nondecreasing in $u_2, \cdots u_p$, has domain $0 \leq x \leq a$, $0 \leq u_i \leq 2b$, $i = 1, \cdots p$, and satisfies *a*) and *b*). Theorem 2 is proved.

When $p = 2, 3$ examples of iterative methods of type (3) are given respectively by the Newton iteration

$$y'_{n+1} = F(x, y_n) + \frac{\partial F(x, y_n)}{\partial y}(y_{n+1} - y_n)$$

$$y_{n+1}(0) = \beta$$

and by the secant method

$$y'_{n+1} = F(x, y_n) + \frac{F(x, y_n) - F(x, y_{n-1})}{y_n - y_{n-1}}(y_{n+1} - y_n)$$

$$y_{n+1}(0) = \beta.$$

Remark. Theorem 2 gives a sufficient condition for the convergence of iterations (3) and may be considered a statement about convergence. Iterative schemes (3) are of practical use, as V. Capra [3] has shown for $p = 2$, when high accuracy in the computation of the solution is required over the whole interval of existence. Furthermore, they may converge faster than the usual Picard method.

Comparison of iterative schemes (3) in terms of rate of convergence, practicality, etc. may constitute further research. The question is also open, whether the local monotonicity of $\varphi(x, u_1, \cdots u_p)$ at $u_i = 0, i = 2, \cdots p$, may be removed for the convergence of iterations (3).

References

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