Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Augustine O. Konnully

An incidence relationship of hyperspheres in E_n

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.4, p. 504–511.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_4_504_0>

Geometria. — *An incidence relationship of hyperspheres in* E_n.

Nota di AUGUSTINE O. KONNULLY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In un iperspazio euclideo sul campo complesso vengono studiati certi sistemi di ipersfere da cui si derivano teoremi e configurazioni generalizzanti quelli di Cox e di Miquel-Clifford relativi a cerchi di un piano.

1. THEOREM. *Let* S_i, $(i = 0, 1, \cdots, n+1)$, *be a set of* $(n+2)$ *hyperspheres in* E_n *having a common orthogonal hypersphere* P. *With every set of* n *of these hyperspheres let a hypersphere distinct from* P *be associated which cuts orthogonally each of the* n *hyperspheres*; *the hypersphere so associated with the set consisting of all the members of the given set of hyperspheres save* S_i *and* S_j *being denoted by* P_{ij}. *Let* S'_k *be the common orthogonal hypersphere of the* $(n+1)$ *hyperspheres* P_{jk}, $(j = 0, 1, \cdots, n+1; j \neq k)$. *Then every set of* $n+2$ *hyperspheres* $S'_h, S'_i, \cdots, S'_m, S_p, S_q, \cdots, S_t$, *all with different subscripts, chosen an even number from* S'_k*'s and the rest from* S_i*'s, has a hypersphere cutting them all orthogonally. In particular, when* n *is even, the hyperspheres* S'_k *have a common orthogonal hypersphere.*

2. Before giving the proof, we first note the condition that, given $n+2$ hyperspheres $S(\vec{a}_i, r_i)$, with $\vec{a}_i$ for centre and r_i for radius, $(i = 0, 1, \cdots \cdots, n+1)$, they have a common hypersphere cutting them all orthogonally. If $L_0, L_1, \cdots, L_{n+2}$ are the cofactors of the elements of the first row of the determinant

$$
(1) \qquad L = \begin{vmatrix}
0 & t_1 & t_2 & \cdots & t_{n+1} & 2 \\
t_1 & \vec{a}_1 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_{n+1} & 1 \\
t_2 & \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdots & \vec{a}_2 \cdot \vec{a}_{n+1} & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
t_{n+1} & \vec{a}_{n+1} \cdot \vec{a}_1 & \vec{a}_{n+1} \cdot \vec{a}_2 & \cdots & \vec{a}_{n+1} \cdot \vec{a}_{n+1} & 1 \\
2 & 1 & 1 & \cdots & 1 & 0
\end{vmatrix}
$$

where $t_i = \vec{a}_i^2 - r_i^2, \cdots, t_{n+1} = \vec{a}_{n+1}^2 - r_{n+1}^2$, then it is easily seen that $S(\vec{a}, r)$, where

$$
(2) \qquad \vec{a} = -\frac{1}{2} L_0^{-1} \left(\sum_{i=1}^{n+1} L_i \vec{a}_i \right),
$$

and

$$
(3) \qquad r = \left(-\frac{1}{4} L_0^{-1} L \right)^{1/2},
$$

(*) Nella seduta del 20 aprile 1974.

represents the common orthogonal hypersphere of the $(n+1)$ hyperspheres $S(\vec{a}_i, r_i)$, $i = 1, 2, \cdots, n+1$, since we have $(\vec{a}_i - \vec{a})^2 = r_i^2 - \frac{1}{4} L/L_0 = r_i^2 + r^2$ for each i.

The hypersphere $S(\vec{a}, r)$ which cuts orthogonally each of the hyperspheres $S(\vec{a}_i, r_i)$, $(i = 1, 2, \cdots, n+1)$, will cut $S(\vec{a}_0, r_0)$ also orthogonally if and only if $r_0^2 + r^2 = (\vec{a}_0 - \vec{a})^2$. The vector $\vec{a}_0$ can be expressed as

$$\vec{a}_0 = \sum_{k=1}^{n+1} g_k^0 \vec{a}_k, \qquad \text{where} \quad \sum_{k=1}^{n+1} g_k^0 = 1; \tag{4}$$

and since

$$(\vec{a}_0 - \vec{a})^2 = \vec{a}_0^2 - 2\sum_{k=1}^{n+1} g_k^0 \vec{a}_k \cdot \vec{a} + \left(-\frac{1}{2} L_0^{-1}\right) \sum_{k=1}^{n+1} L_k \vec{a}_k \cdot \vec{a}$$
$$= \vec{a}_0^2 - \sum_{k=1}^{n+1} g_k^0 t_k - \frac{1}{4} L_0^{-1} L,$$

it means that

$$t_0 = \sum_{k=1}^{n+1} g_k^0 t_k, \qquad (t_i = \vec{a}_i^2 - r_i^2); \tag{5}$$

which expresses the condition—necessary and sufficient condition—that the hyperspheres $S(\vec{a}_0, r_0), S(\vec{a}_1, r_1), \cdots, S(\vec{a}_{n+1}, r_{n+1})$ have a common orthogonal hypersphere.

This condition may be expressed in a more convenient form. If $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are a set of linearly independent vectors, then solving for $g_1^0, g_2^0, \cdots, g_{n+1}^0$ from the equations $\sum_{i=1}^{n+1} g_i^0 \vec{a}_i \cdot \vec{u}_k = \vec{a}_0 \cdot \vec{u}_k$, $k = 1, 2, \cdots, n$, and

$$\sum_{k=1}^{n+1} g_k^0 = 1,$$

the condition reduces to the vanishing of the determinant

$$B = \begin{vmatrix} t_0 & t_1 & \cdots & t_{n+1} \\ \vec{a}_0 \cdot \vec{u}_1 & \vec{a}_1 \cdot \vec{u}_1 & \cdots & \vec{a}_{n+1} \cdot \vec{u}_1 \\ \cdots & \cdots & \cdots & \cdots \\ \vec{a}_0 \cdot \vec{u}_n & \vec{a}_1 \cdot \vec{u}_n & \cdots & \vec{a}_{n+1} \cdot \vec{u}_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \det.(b_0, b_1, \cdots, b_{n+1}), \tag{5'}$$

were $b_k = (t_k, \vec{a}_k \cdot \vec{u}_1, \cdots, \vec{a}_k \cdot \vec{u}_n, 1)$ written as a column.

3. Proof of the theorem

Let $\vec{a}'_k$ be the centre and r'_k, the radius of S'_k, $(k = 0, 1, \cdots, n+1)$. Each vector $\vec{a}'_k$, for $k = 1, 2, \cdots, n+1$, can be expressed as

$$(6) \qquad \vec{a}'_k = g^k_0 \vec{a}'_0 + \sum_{i=1}^{n+1} g^k_i \vec{a}_i , \qquad \text{where } g^k_k = 0 , \qquad \text{and } \sum_{i=0}^{n+1} g^k_i = 1 .$$

Since the hyperspheres $S'_0, S_1, \cdots, S_{k-1}, S'_k, S_{k+1}, \cdots, S_{n+1}$ have a common orthogonal hypersphere, viz., P_{0k}, we have by (5),

$$(7) \qquad t'_k = g^k_0 t'_0 + \sum_{i=1}^{n+1} g^k_i t_i , \qquad (t'_j = \vec{a}'^2_j - r'^2_j) , \qquad \text{for } k = 1, 2, \cdots, n+1 .$$

Further, $S'_i, S'_j, S_0, S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_{j-1}, S_{j+1}, \cdots, S_{n+1}$, $(i \neq 0, j \neq 0)$, have a common orthogonal hypersphere, viz., P_{ij}; the condition for this is, by (5′), the vanishing of the determinant $B(i, j)' = \det. (b_0, b_1, \cdots, b_{i-1}, b'_i, b_{i+1}, \cdots, b'_j, \cdots, b_{n+1})$, where $b'_k = (t'_k, \vec{a}'_k \cdot \vec{u}_1, \vec{a}'_k \cdot \vec{u}_2, \cdots, \vec{a}'_k \cdot \vec{u}_n, 1)$. Substituting for t_0, t'_i, t'_j from (5) and (7) and for $\vec{a}_0, \vec{a}'_i, \vec{a}'_j$ from (4) and (6), it is easily seen that

$B(i, j)' = \pm B(0)' G(0, i, j)$, where $B(0)' = \det (b'_0, b_1, \cdots, b_{n+1})$ and

$$G(0, i, j) = \begin{vmatrix} g^0_0 & g^0_i & g^0_j \\ g^i_0 & g^i_i & g^i_j \\ g^j_0 & g^j_i & g^j_j \end{vmatrix} , \qquad (g^0_0 = 0) .$$

Thus $G(0, i, j) B(0)' = 0$. But $B(0)' \neq 0$, for $B(0)' = 0$ would mean that the hyperspheres $S'_0, S_1, S_2, \cdots, S_{n+1}$ have a common orthogonal hypersphere, which hypersphere would be P, being the common orthogonal hypersphere of the $(n+1)$ hyperspheres $S_1, S_2, \cdots, S_{n+1}$ as also be P_{10}, being the common orthogonal hypersphere of the $(n+1)$ hyperspheres $S'_0, S_2, \cdots, S_{n+1}$, so that P and P_{10} would be identical contrary to the initial choice of P_{10} as distinct from P. Hence $G(0, i, j) = 0$, that is, $g^0_j g^j_i g^i_0 + g^0_i g^i_j g^j_0 = 0$. Thus

$$(8) \qquad g^0_i g^i_j g^j_0 = - g^0_j g^j_i g^i_0 , \qquad (i, j = 1, 2, \cdots, n+1).$$

Now consider any set of hyperspheres $S'_h, S'_i, S'_j, \cdots, S'_m, S_p, S_q, \cdots, S_t$, where $\{h, i, j, \cdots, m\}$ is a subset of the index set $I = \{0, 1, \cdots, n+1\}$ with an even number of elements and $\{p, q, \cdots, t\}$ is its complement in I. The condition for these hyperspheres to have a common orthogonal hypersphere is the vanishing of the determinant $B(h, i, \cdots, m)'$ which is B with its columns of indices $h, i, \cdots, m$ all primed. We shall assume that $h < i < \cdots < m$ and $p < q < \cdots < t$. If $0 \in \{p, q, \cdots, t\}$ we must have $p = 0$ and $h \neq 0$; and then substituting for $t'_h, t'_i, \cdots, t'_m$ and t_0

from (7) and (5), and for $\vec{a}'_h, \vec{a}'_i, \cdots, \vec{a}'_m$ and $\vec{a}_0$ from (6) and (4) we get $B(h, i, \cdots, m)' = \pm B(0)' G(0, h, \cdots, m)$, where

$$G(0, h, \cdots, m) = \begin{vmatrix} g_0^0 & g_h^0 & \cdots & g_m^0 \\ g_0^h & g_h^h & \cdots & g_m^h \\ \cdots & \cdots & \cdots & \cdots \\ g_0^m & g_h^m & \cdots & g_m^m \end{vmatrix}$$

and if $0 \notin \{p, q, \cdots, m\}$, $h = 0$ and $p \neq 0$; and then substituting for $t'_i, t'_j, \cdots, t'_m$ from (7) and for $a'_i, a'_j, \cdots, a'_m$ from (6) we get, $B(h, i, \cdots, m)' = B(0)' G(i, j, \cdots, m)$ where $G(i, j, \cdots, m)$ is the determinant $G(0, h, \cdots, m)$ with its first two rows and columns suppressed. By virtue of (8), on multiplying its columms by $1, g_0^h, g_0^i, \cdots, g_0^m$ respectively and the rows by $-1, g_h^0, g_i^0, \cdots, g_m^0$ respectively, $G(h, i, \cdots, m)$ becomes a skew-symmetric determinant of odd order and therefore vanishes. Similarly $G(i, \cdots, m)$, on its rows being multiplied by $g_i^0, g_j^0, \cdots, g_m^0$ respectively and columns by $g_0^i, g_0^j, \cdots, g_0^m$ becomes a skew–symmetric determinant of odd order and therefore vanishes. Thus in either case $B(h, i, \cdots, m)' = 0$. It follows that $S'_h, S'_i, \cdots, S'_m, S_p, S_q, \cdots, S_t$ have a hypersphere cutting them all orthogonally.

4. Let $U = \{h, i, \cdots, m\}$ and $V = \{p, q, \cdots, t\}$ be complementary subsets of the index set $I = \{0, 1, \cdots, n+1\}$. The set of hyperspheres $S'_h, S'_i, \cdots, S'_m, S_p, S_q, \cdots, S_t$ will have then a common orthogonal hypersphere whenever U has an even number of elements: this hypersphere shall be denoted by $P_{hi\cdots m}$ or $P'_{pq\cdots t}$. The hypersphere cutting orthogonally all the hyperspheres $S_0, S_1, \cdots, S_{n+1}$ has been denoted by P and it arises when U is the null subset of I. As U ranges over all subsets of I with an even number of elements, we get a set of hyperspheres which we shall refer to as P-hyperspheres. And the hyperspheres $S_0, S_1, \cdots, S_{n+1}, S'_0, S'_1, \cdots, S'_{n+1}$ may the refered to as S-hyperspheres. Including P, there will be altogether 2^{n+1} P-hyperspheres. Each P-hypersphere will have $n+2$ S-hyperspheres cutting it orthogonally; and each S-hypersphere will have 2^n P-hyperspheres cutting it orthogonally.

The figure consisting of the 2^{n+1} P-hyperspheres and $2n+4$ S-*hyperspheres* is generated by P and the $n+2$ S-hyperspheres $S_0, S_1, \cdots, S_{n+1}$. The same figure could be thought of equally as generated by any P-hypersphere together with the $n+2$ hyperspheres cutting it orthogonally. In this sense the figure has a homogeneity. Also when n is even, the figure will have a symmetry with $P'_{ij\cdots m}$ counter to $P_{ij\cdots m}$ and S'_i counter to S_i.

5. There are some special cases to be considered.

Let R be the radius of the hypersphere P which cuts orthogonally the hyperspheres $S_0, \cdots, S_{n+1}$. If the hypersphere P_{ij} which we associate with a set of n of these hyperspheres, is to have also radius R, then P_{ij} is unique, since there are two and only two hyperspheres of the same radius R which cut orthogonally the given set of n hyperspheres, of which P is one and P_{ij} is to be distinct from it. We now prove the following

THEOREM. *If the hyperspheres* P_{ij}, $(i , j = 0 , 1 , \cdots , n+1 , i \neq j)$, *have all the same radius as* P, *then every* P-*hypersphere has the same radius as* P. *In particular, when* P *and* P_{ij} *all become points, then every* P-*hypersphere becomes a point.*

Proof. To prove this, it will be enough to show that the hypersphere $P_{12\cdots p}$, p even, has radius R, when P and each P_{ij} have all the same radius R; since, by a relabelling of the hyperspheres S_i, any P-hypersphere could be made to have such a representation.

As the radius r of the common orthogonal hypersphere of $S(\vec{a}_1 , r_1) , \cdots , S(\vec{a}_{n+1} , r_{n+1})$ is given by $r^2 = -\frac{1}{4} L/L_0$, so the radius q of $P_{12\cdots p}$, the common orthogonal hypersphere of $S'_1 , S'_2 , \cdots , S'_p , S_{p+1} , \cdots , S_{n+1}$, is given by $q^2 = -\frac{1}{4} L(1 , 2 , \cdots , p)''/L_0(1 , 2 , \cdots , p)''$, where $L(1 , 2 , \cdots , p)''$ and $L_0(1 , 2 , \cdots , p)''$ are L and L_0 respectively with the rows and columns of indices $1 , 2 , \cdots , p$ all primed, that is,

$$L(1 , 2 , \cdots , p)'' = \begin{vmatrix} 0 & t'_1 & \cdots & t'_p & t_{p+1} & \cdots & t_{n+1} & 2 \\ t'_1 & \vec{a}'_1 \cdot \vec{a}'_1 & \cdots & \vec{a}'_1 \cdot \vec{a}'_p & \vec{a}'_1 \cdot \vec{a}_{p+1} & \cdots & \vec{a}'_1 \cdot \vec{a}_{n+1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t'_p & \vec{a}'_p \cdot \vec{a}'_1 & \cdots & \vec{a}'_p \cdot \vec{a}'_p & \vec{a}'_p \cdot \vec{a}_{p+1} & \cdots & \vec{a}'_p \cdot \vec{a}_{n+1} & 1 \\ t_{p+1} & \vec{a}_{p+1} \cdot \vec{a}'_1 & \cdots & \vec{a}_{p+1} \cdot \vec{a}'_p & \vec{a}_{p+1} \cdot \vec{a}_{p+1} & \cdots & \vec{a}_{p+1} \cdot \vec{a}_{n+1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n+1} & \vec{a}_{n+1} \cdot \vec{a}'_1 & \cdots & \vec{a}_{n+1} \cdot \vec{a}'_p & \vec{a}_{n+1} \cdot \vec{a}_{p+1} & \cdots & \vec{a}_{n+1} \cdot \vec{a}_{n+1} & 1 \\ 2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

and $L_0(1 , 2 , \cdots , p)''$ is the same without its first row and column.

Let M be the determinant

$$D = \begin{vmatrix} 0 & t'_0 & t_1 & t_2 & \cdots & t_{n+1} & 2 \\ t'_0 & \vec{a}'_0 \cdot \vec{a}'_0 & \vec{a}'_0 \cdot \vec{a}_1 & \vec{a}'_0 \cdot \vec{a}_2 & \cdots & \vec{a}'_0 \cdot \vec{a}_{n+1} & 1 \\ t_1 & \vec{a}_1 \cdot \vec{a}'_0 & \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_{n+1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n+1} & \vec{a}_{n+1} \cdot \vec{a}'_0 & \vec{a}_{n+1} \cdot \vec{a}_1 & \vec{a}_{n+1} \cdot \vec{a}_2 & \cdots & \vec{a}_{n+1} \cdot \vec{a}_{n+1} & 1 \\ 2 & 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

bordered in the first row and column by the elements $0, G_0, G_1, \cdots, G_{n+1}, 0$, where $G_{p+1} = G_{p+2} = \cdots = G_{n+1} = 0$ and $G_0, G_1, \cdots, G_p$ are the cofactors of the elements of the first row of the determinant

$$G = \begin{vmatrix} 1 & 0 & 0 \cdots & 0 \\ g_0^1 & g_1^1 & g_2^1 \cdots & g_p^1 \\ g_0^2 & g_1^2 & g_2^2 \cdots & g_p^2 \\ \cdots & \cdots & \cdots & \cdots \\ g_0^p & g_1^p & g_2^p \cdots & g_p^p \end{vmatrix} .$$

Then, it is easily seen that M, on multiplying it twice by G, reduces, by virtue of the relations (6) and (7), to a determinant equal to $-G_0^2 L(1, \cdots, p)''$, that is, $GMG = -G_0^2 L(1, 2, \cdots, p)''$. Since $G = G_0$, we have

$$L(1, 2, \cdots, p)'' = -M = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} D^{ij} G_i G_j ,$$

where

$$D^{ij}, (i, j = -1, 0, 1, \cdots, n+1, n+2),$$

are the cofactors of the elements of D.

Similarly, $L_0(1, 2, \cdots, p)'' = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} D_0^{ij} G_i G_j$, where D_0^{ij}, $(i, j = 0, 1, \cdots \cdots, n+1, n+2)$, are the cofactors of the elements of the determinant D_0 obtained from D on suppressing its first row and first column.

By the formula (3), the square of the radius of $P, P_{01}, \cdots, P_{0n+1}$ will be seen to be given by $-\frac{1}{4} D^{00}/D_0^{00}, -\frac{1}{4} D''/D_0'', \cdots, -\frac{1}{4} D^{n+1,n+1}/D_0^{n+1,n+1}$ respectively. So we have $R^2 = -\frac{1}{4} D^{ii}/D_0^{ii}$, $(i = 0, 1, \cdots, n+1)$. Writing $T_{ij} = D^{ij} + 4R^2 D_0^{ij}$, we obtain

(9) $$T_{ii} = 0, \qquad (i = 0, 1, \cdots, n+1).$$

Also, since the radius of $P_{hk}, (h, k = 1, 2, \cdots, n+1)$, is R, we have $R^2 = -\frac{1}{4} L(h, k)''/L_0(h, k)''$. And it is seen, as above, that

$$L(h, k)'' = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} D^{ij} F_i F_j ; \text{ and } L_0(h, k)'' = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} D_0^{ij} F_i F_j ,$$

where F_0, F_h, F_k are the cofactors of the first row of

$$F = \begin{vmatrix} 1 & 0 & 0 \\ g_0^h & g_h^h & g_k^h \\ g_0^k & g_h^k & g_k^k \end{vmatrix}, \quad \text{and } F_i = 0 \quad \text{for } i \notin \{0, h, k\}.$$

It means, $L(h, k)'' + 4R^2 L_0(h, k)'' = \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} T_{ij} F_i F_j = 0$. Since $T_{ii} = 0$ and $F_i = 0$ for $i \neq 0, h, k$, this reduces to

$$T_{hk} F_h F_k + T_{0h} F_0 F_h + T_{0k} F_0 F_k = 0\,; \quad \text{or} \tag{10}$$

$$T_{hk} = T_{0h} g_k^h / g_0^h + T_{0k} g_h^k / g_0^k\,, \qquad (h, k = 0, 1, 2, \cdots, n+1).$$

It follows that

$$\sum_{k=1}^{p}\sum_{h=1}^{p} G_h T_{hk} G_k = \sum_{h=1}^{p}\sum_{k=1}^{p} G_k g_k^h (1/g_0^h) T_{0h} G_h + \sum_{k=1}^{p}\sum_{h=1}^{p} G_h g_h^k (1/g_0^k) T_{0k} G_k =$$

$$= -\sum_{h=1}^{p} G_0 T_{0h} G_h - \sum_{k=1}^{p} G_0 T_{0k} G_k\,, \quad \text{since} \ \sum_{k=0}^{p} g_k^h G_k = 0\,, \quad (h = 1, 2, \cdots, p).$$

Hence

$$\sum_{h=0}^{p}\sum_{k=0}^{p} G_h T_{hk} G_k = 0\,, \qquad (G_0 T_{00} G_0 = 0, \text{ by } (9)).$$

Thus,

$$L(1, 2, \cdots, p)'' + 4R^2 L_0(1, 2, \cdots, p)'' =$$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} D^{ij} G_i G_j + 4R^2 \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} D^{ij} G_i G_j =$$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} G_i T_{ij} G_j = \sum_{i=0}^{p}\sum_{j=0}^{p} G_i T_{ij} G_j = 0.$$

It follows that

$$R^2 = -\frac{1}{4} L(1, 2, \cdots, p)''/L_0(1, 2, \cdots, p)'' = q^2.$$

That is, the radius of $P_{12\cdots p}$ is R.

6. Of special interest is the case when $R = 0$, that is, P becomes a point so that $S_0, S_1, \cdots, S_{n+1}$ are hyperspheres through a fixed point, and P_{ij} are also points, viz., the points in which the $n + 2$ hyperspheres, takes n by n, meet. For then all the P-hyperspheres become points, giving the relationship: Given $(n + 2)$ hyperspheres $S_0, S_1, \cdots, S_{n+1}$ in E_n, all passing through a fixed point P, with each set of $n + 1$ out of the $n + 2$ hyperspheres if we associate a hypersphere, viz., the one containing the $n + 1$ points in which the $n + 1$ hyperspheres, taken n by n, meet apart from P; and if $S'_0, S'_1, \cdots, S'_{n+1}$ are the hyperspheres so obtained, S'_i being the hypersphere associated with the $n + 1$ hyperspheres of the set excluding S_i; then every set of $n + 2$ hyper-

spheres chosen an even number from S' 's and the rest with different subscripts from S's have a point in common in which they all meet. This relationship is analogous to the Miquel-Clifford configuration of circles and points in a plane.

The $2n + 4$ hyperspheres (S-hyperspheres) and 2^{n+1} points (P-hyperspheres) of which the figure generated will be made up, distribute themselves so that through each point half the number ($= n + 2$) of the hyperspheres pass and on each hypersphere half the number ($= 2^n$) of points lie. Thus it constitutes a configuration of hyperspheres and points in E_n. The $n + 2$ hyperspheres passing through any one of the points of the configuration will generate the same figure.