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On union curves and pseudogeodesics in a Finsler subspace from the standpoint of non-linear connections

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Geometria differenziale. — *On union curves and pseudogeodesics in a Finsler subspace from the standpoint of non-linear connections.* Nota di U. P. SINGH e V. P. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — La teoria delle connessioni non-lineari negli spazi di Finsler è stata studiata da Vagner [1], Barthel [2], Kawaguchi [3] e Singh [4]. Scopo di questa Nota è lo studio di particolari sistemi di curve («Union curves» e pseudogeodetiche) di un sottospazio dello spazio di Finsler.

1. INTRODUCTION

We outline below some fundamental formulae which will be used in the later sections of this paper.

Let X^i be the components of a vector field, g_{ij} the components of the metric tensor and

$$Y_i = g_{ij}(x, X) X^j.$$

Suppose that we are given functions $\overset{1}{\Gamma}{}^i_k(x, X)$, $\overset{2}{\Gamma}_{ik}(x, Y)$ such that the absolute differentials

$$\overset{1}{\delta} X^i = dX^i + \overset{1}{\Gamma}{}^i_k(x, X)\, dx^k, \tag{1.1}$$

$$\overset{2}{\delta} Y_i = dY_i - \overset{2}{\Gamma}_{ik}(x, Y)\, dx^k, \tag{1.2}$$

are respectively the components of a contravariant and covariant vectors. The functions $\overset{1}{\Gamma}{}^i_k(x, X)$, $\overset{2}{\Gamma}_{ik}(x, Y)$ are supposed to be positively homogeneous of first degree in X and Y respectively.

These are used in defining the connection parameter

$$\overset{1}{\Gamma}{}^i_{jk}(x, X) = \frac{\partial \overset{1}{\Gamma}{}^i_k}{\partial X^j} \quad , \quad \overset{2}{\Gamma}{}^i_{jk}(x, Y) = \frac{\partial \overset{2}{\Gamma}_{jk}}{\partial Y_i}. \tag{1.3}$$

We state the following conditions

(A) If X^i undergoes parallel displacement (i.e. $\overset{1}{\delta} X^i = 0$) then so does Y_i (i.e. $\overset{2}{\delta} Y_i = 0$)

This condition is characterised by (Rund [4], p. 238)

$$\overset{2}{\Gamma}_{ik}(x, Y) = \frac{\partial g_{ij}}{\partial x^k} X^j - g_{ij} \overset{1}{\Gamma}{}^j_k(x, X).$$

(B) The connection defined by $\overset{1}{\Gamma}{}^i_k(x, X)$ is metric i.e. the length of the vector X^i remains unchanged under parallel displacement.

(*) Nella seduta del 20 aprile 1974.

2. SUBSPACES OF A FINSLER SPACE

Let $$F_m : \; x^i = x^i(u^\alpha) \qquad (i = 1 \cdots n, \alpha = 1 \cdots n)$$

be a subspace of F_n. The components X^i, X^α of a vector-field of the subspace are related by

$$X^i = B^i_\alpha X^\alpha, \qquad \text{where} \quad B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \tag{2.1}$$

The induced differential $\overset{1}{\delta} X^\alpha$ is defined by

$$\overset{1}{\delta} X^\alpha = B^\alpha_i \overset{1}{\delta} X^i \tag{2.2}$$

where $$B^\alpha_i = g^{\alpha\beta}(u, X)\, g_{ij}(x, X)\, B^j_\beta \quad , \quad g_{\alpha\beta}(u, X)$$

being the metric tensor of F_m. The equation (2.1) yields

$$dX^i = B^i_\beta\, dX^\beta + B^i_{\beta\gamma} X^\beta du^\gamma. \tag{2.3}$$

Defining

$$\overset{1}{\delta} X^\alpha = dX^\alpha + \overset{1}{\Gamma}{}^\alpha_\gamma(u, X)\, du^\gamma \tag{2.4}$$

and using the equations (1.1), (2.2) and (2.3) we find

$$\overset{1}{\Gamma}{}^\alpha_\gamma = B^\alpha_i (B^i_{\beta\gamma} X^\beta + \overset{1}{\Gamma}{}^i_k B^k_\gamma), \tag{2.5}$$

where we use the relations $B^\alpha_i B^i_\beta = \delta^\alpha_\beta$ and

$$dx^k = B^k_\gamma du^\gamma.$$

Assuming that the function $\overset{1}{\Gamma}{}^\alpha_\gamma(u, X)$ is differentiable, we define

$$\overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u, X) = \frac{\partial \overset{1}{\Gamma}{}^\alpha_\gamma(u, X)}{\partial X^\beta}. \tag{2.6}$$

A direct differentiation of the relation

$$B^\alpha_i = g^{\alpha\delta}(u, X)\, g_{ij}(x, X)\, B^j_\delta$$

with respect to X^β will yield (after some simplification)

$$\frac{\partial B^\alpha_i}{\partial X^\beta} = \sum_\mu 2 \underset{\mu}{N}_i \underset{\mu}{M}{}^\alpha_\beta, \tag{2.7}$$

where we used the fact (Rund [5], pag. 158)

$$B^\alpha_i B^j_\alpha = \delta^j_i - \sum_\mu \underset{\mu}{N}{}^j \underset{\mu}{N}_i, \tag{2.8}$$

$\underset{\mu}{N}_i$ being the components of the unit normal vectors,

$$2\, C_{ijk}(x\,,\, X) = \frac{\partial g_{ij}(x\,,\, X)}{\partial X^k} \quad , \quad \underset{\mu}{M}_{\alpha\beta} = C_{ijk}\, B^i_\alpha\, B^j_\beta\, \underset{\mu}{N}^k$$

and

$$\underset{\mu}{M}^\alpha_\beta(u\,,\, X) = g^{\alpha\gamma}\, \underset{\mu}{M}_{\beta\gamma}(u\,,\, X)\,.$$

Differentiating (2.5) with respect to X^β and using the relations (1.3), (2.6), (2.7) and the fact

$$\underset{\mu}{M}^\alpha_\beta(u\,,\, X)\, X^\beta = 0\,,$$

we find

$$\text{(2.9)} \qquad \overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u\,,\, X) = \sum_\mu 2\, \underset{\mu}{N}_i\, \underset{\mu}{M}^\alpha_\beta\, (B^i_{\delta\gamma}\, X^\delta + \overset{1}{\Gamma}{}^i_k\, B^k_\gamma) + B^\alpha_i\, (B^i_{\beta\gamma} + \overset{1}{\Gamma}{}^i_{hk}\, B^h_\beta\, B^k_\gamma)\,.$$

The connection parameter $\overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u\,,\, X)$ is non-symmetric in $\beta\,,\gamma$ and it is positively homogeneous of degree zero in X. This will be called induced « non-linear connection parameters » of the hypersurface.

3. Geodesics of a subspace

From (2.2) and (2.8) we find

$$\text{(3.1)} \qquad \overset{1}{\delta} X^i = B^i_\alpha\, \overset{1}{\delta} X^\alpha + \sum_\mu (\underset{\mu}{N}_j\, \overset{1}{\delta} X^j)\, \underset{\mu}{N}^i.$$

The geodesic of F_n and F_m are given by (Rund [5], p. 240)

$$\text{(3.2)} \qquad \frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih}\, Y_j\, (\overset{1}{\Gamma}{}^j_{hk}\, X^k - \overset{1}{\Gamma}{}^j_h) = 0$$

and

$$\text{(3.3)} \qquad \frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma}\, Y_\beta\, (\overset{1}{\Gamma}{}^\beta_{\gamma\delta}\, X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) = 0$$

respectively $(Y_\beta = g_{\alpha\beta}(u\,,\, X)\, X^\alpha)$.

A calculation based on equations (3.1), (2.9), (2.5) and relations (Rund [5], p. 236),

$$\overset{1}{\Gamma}{}^i_{hk}\, X^h = \overset{1}{\Gamma}{}^i_k \quad , \quad g^{\alpha\gamma}\, B^i_\alpha\, B^h_\gamma = g^{ih} - \sum_\mu \underset{\mu}{N}^i\, \underset{\mu}{N}^h$$

and $\underset{\mu}{M}^\beta_\gamma\, Y_\beta = 0$ gives

$$\text{(3.4)} \qquad \frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih}\, Y_j\, (\overset{1}{\Gamma}{}^j_{hk}\, X^k - \overset{1}{\Gamma}{}^j_h) = B^i_\alpha \left[\frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma}\, Y_\beta\, (\overset{1}{\Gamma}{}^\beta_{\gamma\delta}\, X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma)\right] +$$

$$+ \sum_\mu \left\{\underset{\mu}{N}_j\, \frac{\overset{1}{\delta} X^j}{\delta s} + \underset{\mu}{N}^h\, Y_j\, (\overset{1}{\Gamma}{}^j_{hk}\, X^k - \overset{1}{\Gamma}{}^j_h)\right\} \underset{\mu}{N}^i\,.$$

After differentiating (2.1) we find,

$$\frac{dX^j}{ds} = B^j_{\beta\gamma} X^\beta X^\gamma + B^j_\beta \frac{dX^\beta}{ds} \qquad \left(X^\gamma = \frac{du^\gamma}{ds}\right). \tag{3.5}$$

The equations (1.1), (2.1) and (3.5) and the relation

$$\overset{1}{\Gamma}{}^j_k = \overset{1}{\Gamma}{}^j_{hk} X^h \tag{3.6}$$

give,

$$\underset{\mu}{N}_j \frac{\overset{1}{\delta} X^j}{\delta s} = \underset{\mu}{N}_j (B^i_{\beta\gamma} + \overset{1}{\Gamma}{}^j_{hk} B^h_\beta B^k_\gamma) X^\beta X^\gamma. \tag{3.7}$$

Further the relations (2.1), (3.6) and the fact

$$Y_j = g_{lj}(x, X) X^l$$

yield

$$Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) \underset{\mu}{N}^h = g_{lj} (\overset{1}{\Gamma}{}^l_{hk} - \overset{1}{\Gamma}{}^l_{kh}) B^j_\beta B^k_\gamma \underset{\mu}{N}^h X^\beta X^\gamma. \tag{3.8}$$

After substituting from (3.7) and (3.8) in (3.4), we find

$$\frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) = B^i_\alpha \left[\frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma)\right] + \tag{3.9}$$

$$+ \sum_\rho \underset{\rho}{\Omega}_{\beta\gamma} X^\beta X^\gamma \underset{\rho}{N}^i,$$

where

$$\underset{\mu}{\Omega}_{\beta\gamma} = \underset{\mu}{N}_j (B^j_{\beta\gamma} + \overset{1}{\Gamma}{}^j_{hk} B^h_\beta B^k_\gamma) + g_{lj} (\overset{1}{\Gamma}{}^l_{hk} - \overset{1}{\Gamma}{}^l_{kh}) B^j_\beta B^k_\gamma \underset{\mu}{N}^h.$$

The vectors

$$q^i = \frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) \tag{3.10}$$

and

$$p^\alpha = \frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) \tag{3.11}$$

will be called the first curvature vectors (in V_m, V_n respectively) of the curve C and the scalar $\underset{\mu}{\Omega}_{\beta\gamma}(x, X) X^\beta X^\gamma$ will be called the normal curvature in the direction of the above curve.

4. UNION CURVE AND PSEUDOGEODESICS

We shall obtain the equations of union curve in a subspace of Finsler space equipped with a metric non-linear connection. As proved in [4] it can be shown easily that under this assumption the induced non-linear connection of the subspace is also metric and each of the vectors q^i and p^α is orthogonal to the tangent vector X^i (or X^α) of the curve.

Consider a set of $(n-m)$ congruences of curves in F_n given by the vector fields $\underset{\mu}{\lambda^i}$ $(\mu = m+1 \cdots n)$. At the points of the subspace, we may write

$$\underset{\mu}{\lambda^i} = \underset{\mu}{t^\alpha} B^i_\alpha + \Sigma\, C_{(\mu\nu)} \underset{\nu}{N^i}. \tag{4.1}$$

It is assumed that these vectors are normalised by the condition

$$g_{ij}(x\,,X)\, \underset{\mu}{\lambda^i} \underset{\mu}{\lambda^j} = 1 \tag{4.2}$$

and

$$\det |C_{\mu\nu}| \neq 0\,.$$

A curve of the subspace will be called union curve relative to the congruence $\underset{\mu}{\lambda^i}$ if variety determined by its tangent vector and the first curvature vector q^i contains the vector. Therefore, we have,

$$\underset{\mu}{\lambda^i} = \underset{\mu}{A} X^i + \underset{\mu}{B} q^i. \tag{4.3}$$

This in view of the equations (2.1), (3.9), (3.10), (3.11) and (4.1) and the fact that n vector B^i_α $(\alpha = 1 \cdots m)$, $\underset{\nu}{N^i}$ $(\nu = m+1 \cdots n)$ are linearly independent yields

$$\underset{\mu}{t^\alpha} = \underset{\mu}{A} X^\alpha + \underset{\mu}{B} p^\alpha \tag{4.4}$$

and

$$C_{(\mu\nu)} = \underset{\mu}{B}\, \underset{\nu}{\Omega_{\beta\gamma}} X^\beta X^\gamma \qquad \text{for} \quad \nu = m+1 \cdots n. \tag{4.5}$$

Defining

$$\cos \underset{\mu}{\alpha} = \frac{g_{\alpha\beta}(u\,,X)\, \underset{\mu}{t^\alpha} X^\beta}{\sqrt{g_{\alpha\beta}(u\,,X)\, \underset{\mu}{t^\alpha} \underset{\mu}{t^\beta}}} \qquad \text{and} \quad \underset{\mu}{T} = \sqrt{g_{\alpha\beta}(u\,,X)\, \underset{\mu}{t^\alpha} \underset{\mu}{t^\beta}} \tag{4.6}$$

and using the fact that X^α is orthogonal to p^α we find (from the equation (4.4))

$$\underset{\mu}{A} = \underset{\mu}{T} \cos \underset{\mu}{\alpha}.$$

This relation and equations (4.5), (3.11) reduce (4.4) to

$$\frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma}\, Y_\beta\, (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) = \frac{\underset{\nu}{\Omega_{\beta\gamma}} X^\beta X^\gamma}{C_{(\mu\nu)}} (\underset{\mu}{t^\alpha} - \underset{\mu}{T} \cos \underset{\mu}{\alpha}\, X^\alpha). \tag{4.7}$$

Further in view of (2.4) and the relations $\overset{1}{\Gamma}{}^\beta_\gamma = \overset{1}{\Gamma}{}^\beta_{\delta\gamma} X^\delta$, $X^\alpha = \frac{du^\alpha}{ds}$, the above equations will take the form

$$\frac{dX^\alpha}{ds} + U^\alpha_{\beta\gamma} X^\beta X^\gamma = 0 \tag{4.8}$$

where

$$U^{\alpha}_{\beta\gamma} = \overset{1}{\Gamma}{}^{\alpha}_{\beta\gamma} + g^{\alpha\varepsilon} g_{\delta\beta} (\overset{1}{\Gamma}{}^{\delta}_{\varepsilon\gamma} - \overset{1}{\Gamma}{}^{\delta}_{\gamma\varepsilon}) - \frac{\underset{\nu}{\Omega}_{\beta\gamma}}{C_{(\mu,\nu)}} (\underset{\mu}{t}^{\alpha} - \underset{\mu}{T} \cos \underset{\mu}{\alpha} X^{\alpha}).$$

The equations (4.7) or (4.8) represents a union curve relative to $\underset{\mu}{\lambda}^{i}$.

In order to find the Pseudogeodesics we define a set of $(n-m)$ vectors $\underset{\nu}{Z}^{i} (\nu = m + 1 \cdots n)$ such that

(i) $\underset{\nu}{Z}^{i}$ is a linear combination of $\underset{\nu}{\lambda}^{i}$ and X^{i} and

(ii) it is orthogonal to X^{i}. These conditions may be expressed as

$$\text{(4.9)} \qquad \underset{\nu}{Z}^{i} = \underset{\nu}{C} X^{i} + \underset{\nu}{D} \underset{\nu}{\lambda}^{i} \quad , \quad g_{ij}(x , X) \underset{\nu}{Z}^{i} X^{j} = 0.$$

It is further assumed that the vectors $\underset{\nu}{Z}^{i}$ satisfy the normalising condition

$$\text{(4.10)} \qquad g_{ij}(x , X) \underset{\nu}{Z}^{i} \underset{\nu}{Z}^{j} = 1 .$$

Using the equations (4.1), (4.6), (4.9) and (4.10) we find

$$\text{(4.11)} \qquad \underset{\nu}{C} + \underset{\nu}{D} \operatorname{Cos} \underset{\nu}{\alpha} \underset{\nu}{T} = 0 \quad , \quad \underset{\nu}{D} = (1 - \cos^{2} \underset{\nu}{\alpha} \underset{\nu}{T}^{2})^{-1/2}.$$

Substituting these values in (4.9) and using (4.1) we get,

$$\text{(4.12)} \qquad \underset{\nu}{Z}^{i} = [(\underset{\nu}{t}^{\alpha} - \cos \underset{\nu}{\alpha} \underset{\nu}{T} X^{\alpha}) B^{i}_{\alpha} + \sum_{\rho} C_{(\nu\rho)} \underset{\rho}{N}^{i}] (1 - \cos^{2} \underset{\nu}{\alpha} \underset{\nu}{T}^{2})^{-1/2}.$$

Elimination of $\underset{\rho}{N}^{i}$ from this equation and (3.9) gives

$$\text{(4.13)} \qquad \frac{\overset{1}{\delta} X^{i}}{\delta s} + g^{ih} Y_{j} (\overset{1}{\Gamma}{}^{j}_{hk} X^{k} - \overset{1}{\Gamma}{}^{j}_{h}) = B^{i}_{\alpha} v^{\alpha} +$$

$$+ \sum_{\mu} \sum_{\nu} \underset{\mu}{\Omega}_{\beta\gamma} X^{\beta} X^{\gamma} \bar{C}_{(\nu\mu)} \underset{\nu}{Z}^{i} (1 - \cos^{2} \underset{\nu}{\alpha} \underset{\nu}{T}^{2})^{1/2},$$

where

$$\text{(4.14)} \qquad v^{\alpha} = \frac{\overset{1}{\delta} X^{\alpha}}{\delta s} + g^{\alpha\gamma} Y_{\beta} (\overset{1}{\Gamma}{}^{\beta}_{\gamma\delta} X^{\delta} - \overset{1}{\Gamma}{}^{\beta}_{\gamma}) -$$

$$- \sum_{\mu} \sum_{\nu} \underset{\mu}{\Omega}_{\beta\gamma} (u , X) X^{\beta} X^{\gamma} \bar{C}_{(\nu\mu)} (\underset{\nu}{t}^{\alpha} - \cos \underset{\nu}{\alpha} \underset{\nu}{T} X^{\alpha})$$

and

$$\bar{C}_{(\nu\mu)} = \frac{\text{Cofactor of } C_{(\nu\mu)} \text{ in } |C_{\nu\mu}|}{|C_{(\nu\mu)}|} .$$

The vector v^{α} is called relative first curvature vector of the curve. Since the connection is metric the vector p^{α} is orthogonal to X^{α}. This fact and the equations (3.11), (4.6), (4.14) prove that $g_{\alpha\beta}(x , X) v^{\alpha} X^{\beta} = 0$. Hence

The relative first curvature vector of curve of a Finsler subspace equipped with a metric non-linear connection is orthogonal to the curve.

A curve for which the relative first curvature vector vanishes identically is called pseudogeodesic of the subspace. The curve is given by the equations

$$(4.15)\qquad \frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) - \sum_\mu \sum_\nu \underset{\mu}{\Omega}_{\beta\gamma}(u, X) X^\beta X^\gamma$$

$$\bar{C}_{(\nu\mu)} (\underset{\nu}{t}^\alpha - \cos \underset{\nu}{\alpha} \underset{\nu}{T} X^\alpha) = 0 .$$

Suppose, in particular, that the congruences $\lambda^i_{(\nu)}$ satisfy the conditions

$$(4.16)\qquad \underset{\rho}{\lambda}^i = \underset{\rho}{N}^i \quad \text{for} \quad \rho \neq \mu \quad \text{and} \quad C_{(\mu\mu)} \neq 0 .$$

It is also assumed that the congruence λ^i_μ is consistent with the conditions (refer equation 4.5)

$$(4.17)\qquad \frac{\underset{(\nu)}{\Omega}_{\beta\gamma}(u, X) X^\beta X^\gamma}{C_{(\mu\nu)}} = \frac{\underset{\mu}{\Omega}_{\beta\gamma} X^\beta X^\gamma}{C_{(\mu\mu)}} , \qquad \text{for} \quad \nu = m + 1 \cdots n .$$

Under the condition (4.16) we deduce

$$|C_{(\nu\mu)}| = C_{(\mu\mu)} \neq 0 \quad , \quad \bar{C}_{(\nu\rho)} = \delta_{(\nu\rho)} \quad \text{(Kronecker delta)}$$
$$\text{for all} \quad \nu \neq \mu \quad \text{and} \quad \rho \neq \mu$$

$$\bar{C}_{(\mu\rho)} = 0 \quad \text{for} \quad \rho \neq \mu \quad , \quad \bar{C}_{(\nu\mu)} = -\frac{C_{(\mu\nu)}}{C_{(\mu\mu)}} \qquad \text{for} \quad \nu \neq \mu$$

and

$$\bar{C}_{(\mu\mu)} = \frac{1}{C_{(\mu\mu)}} .$$

The equation (4.15) will now take the form

$$(4.18)\qquad \frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) - \frac{\underset{\mu}{\Omega}_{\beta\gamma} X^\beta X^\gamma}{C_{(\mu\mu)}} (\underset{\mu}{t}^\alpha - \underset{\mu}{T} \cos \underset{\mu}{\alpha} X^\alpha) = 0 .$$

Comparing this with (4.7), it follows that (4.18) represents union curve relative to the congruence $\underset{\mu}{\lambda}^i$. *This proves that a pseudogeodesic is a generalisation of the union curve of the hypersurface.*

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