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ANGELO DI TOMMASO, ANTONIO TRALLI

Reliability tests for structures under general loading processes

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Meccanica. — *Reliability tests for structures under general loading processes* (*). Nota I di ANGELO DI TOMMASO e ANTONIO TRALLI, presentata (**) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — La programmazione matematica e il metodo degli elementi finiti sono utilizzati in questo lavoro allo scopo di istituire un test di affidabilità per strutture sottoposte ad un generale processo di carico. Vengono a tal fine considerati elementi a sforzo costante ed a comportamento elastico lineare; in tal modo è possibile correlare il dominio delle azioni esterne a quello delle risposte strutturali.

Nella Nota I vengono costruiti i modelli matematici, nella Nota II si istituiscono i tests di affidabilità.

1. INTRODUCTION

In an actual structure subjected to a general loading process it is quite impossible to evaluate exactly the amount and position of every *external action* (i.e. forces, dislocations, assigned displacements...); further these external actions very often can be considered *statically* time-depending.

A complete *Reliability test* must take into account the set of all *structural responses* (i.e. displacements, internal stresses, reactions of constraints) produced by the set of all possible *external actions*.

The first problem is to define a *mathematical model of the structure* and a *mathematical model of external actions*. As is well known, a mathematical model of the structure can be defined through the *finite element idealization* and the constitutive laws of the material. We note that a very consistent model of external actions is represented by a set of inequalities (geometrically representing a domain); in such a way it is possible to take into account the indeterminations, the statistical nature and the time-dependence of actual external actions.

A much hard work is needed in order to perform a complete *Reliability test* taking into account the whole set of external actions if we wish to use the influence coefficient method.

Many efforts have been made in these years using mathematical programming connected to finite element method in developing advanced methods of structural mechanics ([6], [7], [9], Plastic Analysis; [8] Structural Optimization; [5], [10] Shakedown theory).

Our purpose in this paper is to suggest an automatic procedure suitable for performing a complete *Reliability test* for structures with linear behaviour, using *mathematical programming* and *finite element* idealization.

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2. Structural discrete model and basic relations

A well known approach consists of developing a discrete model of the structure by finite element method.

Let us consider m elements connected by n nodes; we assume as kinematic variables the three components of displacements in x, y, z directions, as static variables the three components of external forces acting on the nodes in x, y, z directions.

It is convenient to consider a partitioning of displacements column matrix $\boldsymbol{u}$ in conjunction with the forces column matrix $\mathbf{F}$ (in the same order as the displacements in vector $\boldsymbol{u}$):

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_\alpha \\ \text{-----} \\ \boldsymbol{u}_\beta \end{bmatrix}^{3n} \qquad \mathbf{F} = \begin{bmatrix} \mathbf{F}_\alpha \\ \text{-----} \\ \mathbf{F}_\beta \end{bmatrix}^{3n} .$$

where $\boldsymbol{u}_\beta$ is the β-subvector of the assigned displacements and $\boldsymbol{u}_\alpha$ is the α-subvector of the free (unknown) displacements; then $\mathbf{F}_\beta$ is the β-subvector of the forces corresponding to the $\boldsymbol{u}_\beta$ displacements (reactions of constraints) (1) and $\mathbf{F}_\alpha$ is the α-subvector of the assigned forces.

Therefore the problem starts from the definition of a *load condition vector* $\mathbf{I}$ containing, as components, the assigned external forces $\mathbf{F}_\alpha$, the assigned dislocations in each element Δ and the assigned displacements $\boldsymbol{u}_\beta^0$:

$$\mathbf{I} = \mathbf{I}\,(\mathbf{F}_\alpha\,,\,\Delta\,,\,\boldsymbol{u}_\beta^0) \tag{1}$$

and it consists of the search of the *structural response vector* $\mathbf{I}^*$ containing as components, the unknown displacements $\boldsymbol{u}_\alpha$, the internal stresses (or internal generalized forces) $\mathbf{Q}$ and the reactions of constraints $\mathbf{F}_\beta$:

$$\mathbf{I}^* = \mathbf{I}^*\,(\boldsymbol{u}_\alpha\,,\,\mathbf{Q}\,,\,\mathbf{F}_\beta)\,. \tag{2}$$

Constant stress elements [1], [2], [8], it seems a convenient choice for our purposes; we develop the mathematical model with reference to tetrahedral elements whose interaction with the surrounding elements is realized through the nodes (vertexes) only. A quite obvious particularization can be obtained for triangular elements in plane stress problems or for pin-joined bars.

To each constant stress j-element the vectors (2):

$$\text{stress vector } \tilde{\sigma}^j = [\sigma_x^j\,\sigma_y^j\,\sigma_z^j\,\tau_{xy}^j\,\tau_{xz}^j\,\tau_{yz}^j] \tag{3}$$

$$\text{strain vector } \tilde{\varepsilon}^j = [\varepsilon_x^j\,\varepsilon_y^j\,\varepsilon_z^j\,\gamma_{xy}^j\,\gamma_{xz}^j\,\gamma_{yz}^j] \tag{4}$$

can be univocally associated.

(1) Here and in what follows we use the term *displacements* or *forces* instead of *components of displacement* or *components of force* in the x, y, z directions.

(2) A superposed tilde ($\sim$) denotes transposition.

Let us introduce the *natural generalized strains* representing the six side elongations of the j-tetrahedron (fig. 1):

$$\tilde{q}^j = [q_1^j \, q_2^j \, q_3^j \cdots q_6^j] \tag{5}$$

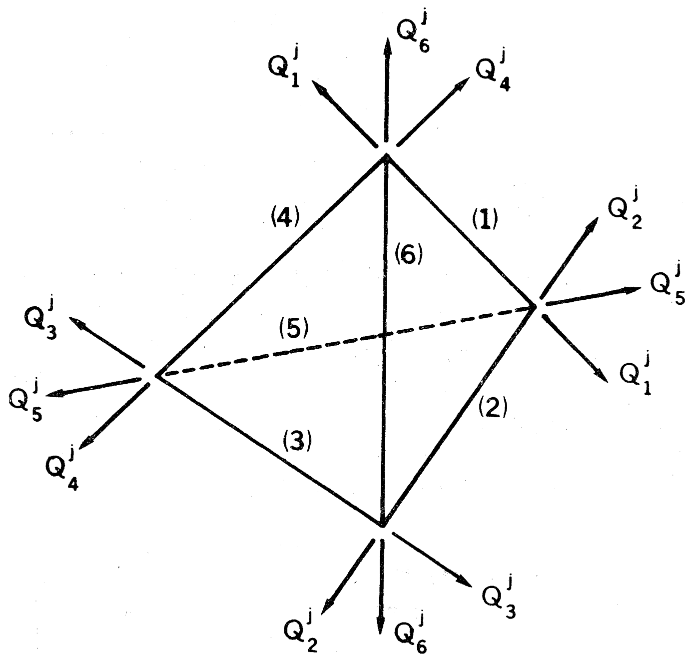

Fig. 1.

Obviously the linear non-singular transformation follows:

$$\boldsymbol{q}^j = \mathbf{T}^j \, \varepsilon^j \tag{6}$$

where $\mathbf{T}^j$ is a square matrix (6×6) whose terms depend on the geometry of the j-element (coordinates of the nodes).

If any anelastic dislocation exists in the j-element, we have for the strain vector:

$$\varepsilon^j = (\varepsilon^{jE} + \varepsilon^{jE\Delta}) + \varepsilon^{j\Delta} \tag{7}$$

where:

$\varepsilon^{j\Delta}$ represents the *anelastic* strain vector due to dislocation Δ^j;

$\varepsilon^{jE\Delta}$ represents the *elastic* strain vector corresponding to anelastic one;

ε^{jE} represents the *elastic* strain vector due only to external forces $\mathbf{F}_\alpha$, to reactions $\mathbf{F}_\beta$ and to assigned displacements $\boldsymbol{u}_\beta^0$.

From (6) and (7) it follows:

$$\boldsymbol{q}^j = \mathbf{T}^j \, (\varepsilon^{jE} + \varepsilon^{jE\Delta}) + \mathbf{T}^j \, \varepsilon^{j\Delta} \tag{8}$$

or:

$$q^j = e^j + \Delta^j \tag{9}$$

with:

$$e^j = \mathbf{T}^j (\varepsilon^{jE} + \varepsilon^{jE\Delta}) \tag{10}$$

$$\Delta^j = \mathbf{T}^j \varepsilon^{j\Delta}. \tag{11}$$

It is clear that $\tilde{e}^j = (e_1 e_2 \cdots e_6)$ represents the *elastic* elongations of the sides due to elastic strains (corresponding to external forces $\mathbf{F}_\alpha$, to reactions $\mathbf{F}_\beta$ and internal dislocations Δ); Δ^j represents the *anelastic* elongations of the sides.

Then it is possible to define a *natural generalized internal forces vector* [2] referred to the j-element:

$$\tilde{\mathbf{Q}}^j = [Q_1^j Q_2^j \cdots Q_6^j] \tag{12}$$

by the principle of virtual work:

$$\tilde{\mathbf{Q}}^j q^j = \int_{V^j} \sigma^j \varepsilon^j \, dV \tag{13}$$

where V^j represents the volume of the j-element.

From (6) and for a constant stress element we have:

$$\mathbf{Q}^j = V^j (\tilde{\mathbf{T}}^j)^{-1} \sigma^j. \tag{14}$$

From the constitutive law:

$$\sigma^j = \chi (\varepsilon^{jE} + \varepsilon^{jE\Delta}) \tag{15}$$

and from (14), (15) and (10) it follows:

$$\mathbf{Q}^j = \mathbf{D}^j e^j \tag{16}$$

with:

$$\mathbf{D}^j = V^j (\tilde{\mathbf{T}}^j)^{-1} \chi (\mathbf{T}^j)^{-1}$$

that is the *stiffness matrix* of j-element.

Assembling the m elements we define the following super-vectors:

$$\tilde{\mathbf{Q}} = [\mathbf{Q}^1 \mathbf{Q}^2 \cdots \mathbf{Q}^j \cdots \mathbf{Q}^m] \tag{17.a}$$

$$\tilde{e} = [q^1 q^2 \cdots q^j \cdots q^m] \tag{17.b}$$

$$\tilde{q} = [e^1 e^2 \cdots e^j \cdots e^m] \tag{17.c}$$

$$\tilde{\Delta} = [\Delta^1 \Delta^2 \cdots \Delta^j \cdots \Delta^m]. \tag{17.d}$$

In this way we have:

$$\mathbf{Q} = \mathbf{D}\boldsymbol{e} = \mathbf{D}(\boldsymbol{q} - \Delta) \tag{18.a}$$

taking into account that from (9) and (17.b), (17.c), (17.d) it follows $\boldsymbol{e} = \boldsymbol{q} - \Delta$; the matrix $\mathbf{D}$ is a block matrix: $\mathbf{D} = \text{diag}\,[\mathbf{D}^1 \cdots \mathbf{D}^m]$.

Compatibility throughout the discretized structure requires [1], [2]:

$$\mathbf{B}\boldsymbol{u} = \boldsymbol{q} \tag{19}$$

where $\mathbf{B}$ is a matrix that depends only on the geometric layout of the model. Taking into account the partitioning over $\boldsymbol{u}$ we have:

$$\boldsymbol{q} = [\mathbf{B}_\alpha \mid \mathbf{B}_\beta] \begin{bmatrix} \boldsymbol{u}_\alpha \\ \hdashline \boldsymbol{u}_\beta \end{bmatrix}. \tag{20}$$

From (18.a)

$$\boldsymbol{q} = \mathbf{D}^{-1}\mathbf{Q} + \Delta. \tag{18.b}$$

Then compatibility equation (19) can be written:

$$\mathbf{B}\boldsymbol{u} - \mathbf{D}^{-1}\mathbf{Q} = \Delta. \tag{21}$$

Equilibrium equations are [1], [2], [3]:

$$\tilde{\mathbf{B}}\mathbf{Q} = \mathbf{F}. \tag{22}$$

Taking into account the partitioning over $\mathbf{B}$ [see (20)], the equilibrium (22) and compatibility (21) equations are:

$$\begin{cases} \begin{bmatrix} \tilde{\mathbf{B}}_\alpha \\ \hdashline \tilde{\mathbf{B}}_\beta \end{bmatrix} \mathbf{Q} = \begin{bmatrix} \mathbf{F}_\alpha \\ \hdashline \mathbf{F}_\beta \end{bmatrix} \\ [\mathbf{B}_\alpha \mid \mathbf{B}_\beta] \begin{bmatrix} \boldsymbol{u}_\alpha \\ \hdashline \boldsymbol{u}_\beta \end{bmatrix} - \mathbf{D}^{-1}\mathbf{Q} = \Delta \end{cases} \tag{23.a}$$

This system (23) following the form used by Di Pasquale for space pin-jointed trusses [3] can be arranged in this way:

$$\left[\begin{array}{cc:c} \multicolumn{2}{c:}{\mathbf{0}} & \tilde{\mathbf{B}}_\alpha \\ & & \tilde{\mathbf{B}}_\beta \\ \hdashline \mathbf{B}_\alpha & \mathbf{B}_\beta & -\mathbf{D}^{-1} \end{array}\right] \begin{bmatrix} \boldsymbol{u}_\alpha \\ \hdashline \boldsymbol{u}_\beta \\ \hdashline \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\alpha \\ \hdashline \mathbf{F}_\beta \\ \hdashline \Delta \end{bmatrix}. \tag{23.b}$$

For our purpose, in order to obtain the vector $\mathbf{I}$ and $\mathbf{I}^*$ previously defined in (1) and (2) it is necessary to introduce the unknown forces (reactions of constraints) $\mathbf{F}_\beta$ in the unknown vector (« *structural response vector* $\mathbf{I}^*$ ») and the assigned displacement $\boldsymbol{u}_\beta$ in the data vector (« *load condition vector* $\mathbf{I}$ »). In order to obtain this result we put [3] the system (23.b) in the form [4]:

$$\begin{bmatrix} \mathbf{o} & \mathbf{o} & \widetilde{\mathbf{B}}_\alpha & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \widetilde{\mathbf{B}}_\beta & \mathbf{I} \\ \mathbf{B}_\alpha & \mathbf{B}_\beta & -\mathbf{D}^{-1} & \mathbf{o} \\ \mathbf{o} & \mathbf{I} & \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_\alpha \\ \boldsymbol{u}_\beta \\ \mathbf{Q} \\ -\mathbf{F}_\beta \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\alpha \\ \mathbf{o} \\ \Delta \\ \boldsymbol{u}_\beta^0 \end{bmatrix}. \tag{24.a}$$

Then the general formulation of the problem, more compactly, can be written:

$$\mathbf{A}\mathbf{I}^* = \mathbf{I} \tag{25}$$

or:

$$\mathbf{I}^* = \mathbf{A}^{-1}\,\mathbf{I} \tag{26}$$

being $\mathbf{A}$ the square matrix (if non singular) appearing in the (24.a) and

$$\widetilde{\mathbf{I}} = [\widetilde{\mathbf{F}}_\alpha \,\vdots\, \widetilde{\mathbf{o}} \,\vdots\, \widetilde{\Delta} \,\vdots\, \widetilde{\boldsymbol{u}}_\beta^0] \tag{27}$$

$$\widetilde{\mathbf{I}}^* = [\widetilde{\boldsymbol{u}}_\alpha \,\vdots\, \widetilde{\boldsymbol{u}}_\beta \,\vdots\, \widetilde{\mathbf{Q}} \,\vdots\, -\widetilde{\mathbf{F}}_\beta] \tag{28}$$

the *load condition vector* and *structural response vector* respectively.

The matrix $\mathbf{A}^{-1}$ can be found inverting the matrix $\mathbf{A}$; one has it results:

$$\mathbf{A}^{-1} = \begin{bmatrix} (\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1} & \mathbf{o} & (\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D} & -(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\beta \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{I} \\ \mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1} & \mathbf{o} & \mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D} - \mathbf{D} & -\mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\beta + \mathbf{D}\mathbf{B}_\beta \\ -\widetilde{\mathbf{B}}_\beta\mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1} & \mathbf{I} & -\widetilde{\mathbf{B}}_\beta\mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D} + \widetilde{\mathbf{B}}_\beta\mathbf{D} & \widetilde{\mathbf{B}}_\beta\mathbf{D}\mathbf{B}_\alpha(\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\alpha)^{-1}\widetilde{\mathbf{B}}_\alpha\mathbf{D}\mathbf{B}_\beta - \widetilde{\mathbf{B}}_\beta\mathbf{D}\mathbf{B}_\beta \end{bmatrix} \tag{29}$$

As it can be observed the inversion of the whole matrix $\mathbf{A}$ implies only the usual computational work of inverting the external stiffness $(\widetilde{\mathbf{B}}_\alpha\,\mathbf{D}\mathbf{B}_\alpha)$. Then we note that if $(\widetilde{\mathbf{B}}_\alpha\,\mathbf{D}\mathbf{B}_\alpha)$ is singular $\mathbf{A}$ is also singular as it appears from

(3) The matrix **o** is a matrix whose elements are all 0 and **I** is the identity (matrix whose diagonal elements are all 1 while the other elements are zero).

(29); but, as is well known, this singularity means that either some parts of structures form a mechanism, or rigid body degrees of freedom are not constrained properly by constraints.

3. A DEFINITION OF RELIABILITY FOR THE DISCRETIZED STRUCTURE

The most general *load process* can be represented by a *set* $\mathcal{T}$ of all *load conditions vectors* **I** acting on the structure during it life.

As it shown in (26):

$$\forall \mathbf{I} \Rightarrow \mathbf{I}^* = \mathbf{A}^{-1}\,\mathbf{I} \tag{30}$$

therefore it is possible to obtain the set:

$$\mathcal{T}^*\,\{\mathbf{I}^* \mid \mathbf{I}^* = \mathbf{A}^{-1}\,\mathbf{I}\,,\quad \mathbf{I} \in \mathcal{T}\} \tag{31}$$

that obviously represents the set of all *structural response vectors* $\mathbf{I}^*$.

If we define $\mathcal{E}^*$ as the set of all *admissible structural response vectors* we say that $\mathcal{T}$ is an *admissible load process* if and only if:

$$\mathcal{T}^* \subseteq \mathcal{E}^* \tag{32}$$

and this settles the *Reliability* for the discretized structure or for the structural discrete model (see fig. 2).

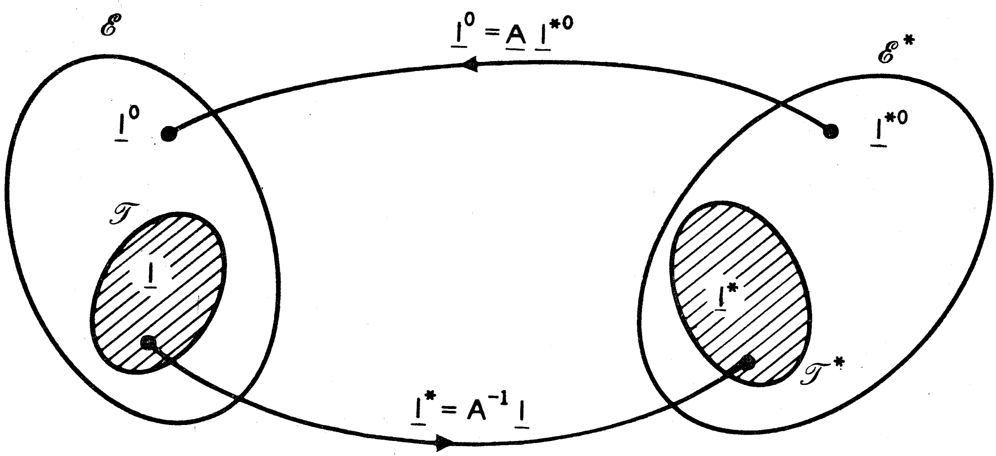

Fig. 2.

As an alternative procedure it is possible to define:

$$\mathcal{T}\,\{\mathbf{I}^0 \mid \mathbf{I}^0 = \mathbf{A}\mathbf{I}^{*0},\ \mathbf{I}^{*0} \in \mathcal{E}^*\} \tag{33}$$

that is the set of all *admissible load condition vectors.*

In this case the condition

$$\mathcal{T} \subseteq \mathcal{E} \tag{34}$$

settles the *Reliability* for the discretized structure or for the structural discrete model under the $\mathcal{T}$ *loading process.*

4. Mathematical model of a general loading process

The definition of the set of all load conditions vectors $\mathscr{T}$ depends on the required functions of the structure, on the defects of assembling, on the thermal effects, on the settlements of foundations, etc.

A reasonable simplification is to assign $\mathscr{T}$ by a piecewise linear domain characterized, as usual, by a matrix $\mathbf{V}$ and vector $\mathbf{L}$; so we can write:

$$\mathscr{T}\{\mathbf{I} \mid \tilde{\mathbf{I}} = [\tilde{\mathbf{F}}_\alpha \mid \tilde{\mathbf{o}} \mid \tilde{\Delta} \mid \tilde{\boldsymbol{u}}^0_\beta] \, , \, \mathbf{VI} \leq \mathbf{L}\} \, . \tag{35}$$

In many cases the assigned forces $\mathbf{F}_\alpha$, the dislocations and the assigned displacements are not joined each other, then it results:

$$\mathbf{V} = \text{diag} \; [\mathbf{V}^{\mathrm{F}} , \mathbf{V}^{\Delta} , \mathbf{V}^{u}] \, . \tag{36}$$

Consequently it follows $\mathscr{T}^*$ set of all structural response vectors from (26), (31), (35):

$$\mathscr{T}^* \{\mathbf{I}^* \mid \tilde{\mathbf{I}}^* = [\tilde{\boldsymbol{u}}_\alpha \, , \tilde{\boldsymbol{u}}_\beta \, , \tilde{\mathbf{Q}} \, , -\tilde{\mathbf{F}}_\beta] \, , \; \mathbf{VAI}^* \leq \mathbf{L}\} \, . \tag{37}$$

5. Mathematical model of the set of an admissible structural response

The definition of the set of all admissible structural response vectors $\mathscr{E}^*$ depends on the flexibility requirements for the structure, on the mechanical properties of the material, on the characteristics of constraints, etc. Again it is reasonable to assign $\mathscr{E}^*$ by a piecewise linear domain characterized by a matrix $\mathbf{N}$ and vector $\mathbf{M}$; so:

$$\mathscr{E}^* \{\mathbf{I}^* \mid \tilde{\mathbf{I}}^* = [\tilde{\boldsymbol{u}}_\alpha \mid \tilde{\boldsymbol{u}}_\beta \mid \tilde{\mathbf{Q}} \mid -\tilde{\mathbf{F}}_\beta] \, , \, \mathbf{NI}^* \leq \mathbf{M}\} \, . \tag{38}$$

In many cases we have independent conditions on the displacements $\boldsymbol{u}$, on the stresses $\mathbf{Q}$, on the constraints $\mathbf{F}_\beta$, so:

$$\mathbf{N} = \text{diag} \; [\mathbf{N}^{u} , \mathbf{N}^{\mathrm{Q}}, \mathbf{N}^{\mathrm{F}}] \, . \tag{39}$$

The domain $\mathscr{E}$ of all admissible load condition vectors will be from (26), (33), (38):

$$\mathscr{E}\{\mathbf{I} \mid \tilde{\mathbf{I}} = [\tilde{\mathbf{F}}_\alpha \mid \tilde{\mathbf{o}}_\beta \mid \tilde{\Delta} \mid \tilde{\boldsymbol{u}}^0_\beta \,], \, \mathbf{NA}^{-1} \, \mathbf{I} \leq \mathbf{M}\} \, . \tag{40}$$

Concerning the mechanical properties of material in developing $\mathscr{E}^*$ domain obviously it is possible to consider the yield piecewise surface of every element. (In that follows we will indicate with $\mathscr{E}^{*y}$ this yield domain).

If we wish to use the domain $\mathscr{E}^*$ in the sense of *Admissible stress method* we have to introduce a safety factor on the yielding point of uniaxial traction test, and this is the traditional way.

REFERENCES

[1] O. C. ZIENKIEWICZ and Y. K. CHEUNG (1967) – *The Finite Element Method in Structural and Continuum Mechanics*, McGraw–Hill, London.

[2] J. H. ARGYRIS (1965) – *Continua and Discontinua, Proceedings of a Conference on Matrix Methods in Structural Mechanics*. Wright–Patterson Air Force Base, Ohio.

[3] S. DI PASQUALE (1971) – *Metodi di Calcolo per le Structure Spaziali*, Progettare Acciaio, Vol. I, Collana Italsider, Genova.

[4] S. DI PASQUALE and S. BRICCOLI (1973) – *Applicazione di un Metodo Generale di Calcolo alle Travi ad Asse Curvilineo*, « Costruzioni metalliche », *3*.

[5] G. MAIER (1969) – *Shake–Down Theory in Perfect Elastoplasticity with Associated and Nonassociated Flow–Laws: a Finite Element, Linear Programming Approach*, « Meccanica », *3*,

[6] G. MAIER (1971) – *Incremental Elastoplastic Analysis in the Presence of Large Displacements and Physical Instabilizing Effects*, « Int. J. Solids Struct. », *7*, 343–372.

[7] E. ANDERHEGGEN and H. KNÖPFEL (1972) – *Finite Element Limit Analysis Using Linear Programming*, « Int. J. Solids Structures O », 1413–1431.

[8] G. MAIER, A. ZAVELANI–ROSSI and D. BENEDETTI (1972) – *A finite Element Approach to Optimal Design of Plastic Structure*, « Int. J. num. Meth. Eng. », *4*, (4).

[9] E. FACCIOLI and E. VITIELLO (1973) – *A finite Element, Linear Programming Method for Limit Analysis of thin Plates*, « Int. J. num. Math. Engng. », *5* (3).

[10] G. MAIER (1972) – *A Shake–Down Matrix Theory allowing for Workhardening and Second–Order Geometric Effects*, Symposium on Foundations of Plasticity, Warsaw.