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Some Partitions of a Rectangular Matrix

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Algebra. — *Some Partitions of a Rectangular Matrix.* Nota di A. DUANE PORTER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si ottengono espressioni esplicite per il numero delle partizioni di una matrice B su di un campo finito quale somma di matrici di dato tipo, come ad esempio $B = U_2 U_1 A + DV_1 V_2$ con le A e D matrici assegnate e le U, V soggette a condizioni di tipo dato.

1. INTRODUCTION

Let A be an $m \times n$ matrix of rank r_1, D be an $s \times t$ matrix of rank r_2, and B be an $s \times n$ matrix. In [2] and [3] John H. Hodges found the number of solutions in a finite field of the matrix equation $UA + DV = B$, where U is $s \times m$ and V is $t \times n$. Certain generalizations of this problem are possible. In particular, one can discuss the number of partitions of a matrix B as defined by

$$(1.1) \qquad U_1 \cdots U_\alpha A + DV_1 \cdots V_\beta = B,$$

where A, B, D are as defined above and U_i, V_j, $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$ are matrices of arbitrary sizes subject to the condition that product sum and equality are defined. This number is discussed in Theorem 1 subject to certain restrictions on A and D. These restrictions can be removed if somewhat easier to handle partitions of the form

$$(1.2) \qquad U_1 \cdots U_\alpha A X_1 \cdots X_\gamma + Y_1 \cdots Y_\delta D V_1 \cdots V_\beta = B$$

are discussed. Partitions of the form (1.2) have already been considered in [7]. This paper is the analog for rectangular matrices of a paper written by the author concerning skew matrices [6]. Later in this paper we find the number of partitions of a matrix B into a sum of k matrices where each is in the form of the left side of (1.1).

2. NOTATION AND PRELIMINARIES

Let $F = GF(q)$ be the finite field of $q = p^f$ elements, p odd. Matrices with elements from F will be denoted by Roman capitals A, B, $\cdots$. $A(s, m)$ will denote a matrix of s rows and m columns and $A(s, m; r)$ a matrix of the same dimensions with rank r. I_r will denote the identity matrix of order r and $I(s, m; r)$ will denote an $s \times m$ matrix with I_r in its upper left hand

(*) Nella seduta del 28 maggio 1974.

corner and zeros elsewhere. If $A = (a_{ij})$ is $n \times n$, then $\sigma(A) = a_{11} + \cdots + a_{nn}$ will be called the trace of A. It is apparent that $\sigma(A+B) = \sigma(A) + \sigma(B)$ and for A, B square $\sigma(AB) = \sigma(BA)$.

For $a \in F$, we define

$$(2.1) \qquad e(a) = \exp 2\pi i t(a)/p \quad ; \quad t(a) = a + a^p + \cdots + a^{p^{f-1}},$$

where by its definition $t(a) \in GF(p)$. It follows directly from (2.1) that

$$(2.2) \qquad e(a+b) = e(a)\,e(b) \ , \quad \sum_b e(ab) = \begin{cases} q \ , & a = 0 \ , \\ 0 \ , & a \neq 0 \ , \end{cases}$$

where the sum is over all $b \in F$.

A direct application of (2.1) and the definition of trace also will show that if $A = A(m, n)$, then

$$(2.3) \qquad \sum_B e\{\sigma(AB)\} = \begin{cases} q^{mn} \ , & A = 0 \ , \\ 0 \ , & A \neq 0 \ , \end{cases}$$

where in this case the sum is over all matrices $B = B(n, m)$. The number $g(a, b; y)$ of $a \times b$ matrices of rank y is given by Landsberg [4] to be

$$(2.4) \qquad g(a, b; y) = q^{y(y-1)/2} \prod_{i=1}^{y} (q^{a-i+1} - 1)(q^{b-i+1} - 1)/(q^i - 1).$$

Following [1; 8.4], if $B = B(s, t; \rho)$, we define

$$(2.5) \qquad H(B, z) = \sum_{C(t,s;z)} e\{-\sigma(B, z)\},$$

where the sum is over all matrices $C(t, s; z)$. This sum is evaluated [1; Theor. 7] to be

$$(2.6) \qquad H(B, z) = q^{\rho z} \sum_{j=0}^{z} (-1)^j q^{j(j-2\rho-1)/2} \begin{bmatrix} \rho \\ j \end{bmatrix} g(s-\rho, t-\rho; z-j),$$

with

$$\begin{bmatrix} \rho \\ j \end{bmatrix} = (1-q^\rho) \cdots (1-q^{\rho-j+1})/(1-q) \cdots (1-q^j).$$

3. An evaluation of (1.1)

We are now able to prove

Theorem 1. *Let* α, β *be integers* ≥ 2; $A = A(m, n; n)$; $D = D(s, t; s)$; $B = B(s, n; \rho)$; $U_1 = U_1(s, s_1)$, $U_i = U_i(s_{i-1}, s_i)$ *for* $1 \leq i < \alpha$; $U_\alpha = U_\alpha(s_{\alpha-1}, m)$; $V_1 = V_1(t, t_1)$; $V_j = V_j(t_{j-1}, t_j)$ *for* $1 \leq j < \beta$; $V_\beta = V_\beta(t_{\beta-1}, n)$, *where* $m, n, s, t, \rho, s_1, \cdots, s_{\alpha-1}, t_1, \cdots, t_{\beta-1}$ *represent arbitrary positive integers. Then the number* N *of partitions of a matrix* B *as described in* (1.1) *is given by*

$$N = q^{r-st} \sum_{z=0}^{(n,s)} H(B, z)\, N_\alpha(z)\, N_\beta(z),$$

where $r = ms_{\alpha-1} + nt_{\beta-1}$; $H(B, z)$ *is given by* (2.5) *and* (2.6); (n, s) = *minimum of* n *and* s; $N_\alpha(z)$ *is given by* (3.4) *and* $N_\beta(z)$ *is obtained from* $N_\alpha(z)$ *by replacing* α *with* β *and* s_k *with* t_k.

Proof. By noting (2.3), we may express the number of partitions of B as described by (1.1) by the following formula:

$$N = q^{-sn} \sum_{C} S(U_1, \cdots, U_\alpha, V_1, \cdots, V_\beta)\, e\{\sigma([U_1 \cdots U_\alpha A + DV_1 \cdots V_\beta - B]\, C)\},$$

where $S(U_1, \cdots, U_\alpha, V_1, \cdots, V_\beta)$ denotes a summation over all U_i, V_j, $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$ as these matrices are defined above, and the sum over C is over all $C = C(n, s)$. If we divide the sum over C into successive sums over $C(n, s; z)$, $0 \leq z \leq (n, s)$ = minimum of n and s, note (2.2) as well as the properties of trace, we may write the above equation as

$$\text{(3.1)} \qquad N = q^{-sn} \sum_{z=0}^{(n,s)} \sum_{C(n,s;z)} e\{-\sigma(BC)\} \cdot$$
$$S(U_1, \cdots, U_\alpha, V_1, \cdots, V_\beta)\, e\{\sigma(U_1 \cdots U_\alpha AC)\}\, e\{\sigma(DV_1 \cdots V_\beta C)\}.$$

However, since the variable matrices in each of the exponential functions in the second line above are distinct from each other, we may write this line as

$$\text{(3.2)} \quad S(U_1, \cdots, U_\alpha)\, e\{\sigma(U_1 \cdots U_\alpha AC)\}\, S(V_1, \cdots, V_\beta)\, e\{\sigma(DV_1 \cdots V_\beta C)\}.$$

We must now evaluate each sum in (3.2). To do this we first note that for any fixed choice of $U_1, \cdots, U_\alpha, V_1, \cdots, V_\beta$ and any $C = C(n, s; z)$, we have $\sigma(U_1 \cdots U_\alpha AC) = \sigma(ACU_1 \cdots U_\alpha)$ and $\sigma(DV_1 \cdots V_\beta C) = \sigma(CDV_1 \cdots V_\beta)$. By making these substitutions into (3.2) and summing over U_α and V_β in accordance with (2.3), we can see that the only nonzero contributions to (3.2) come from terms such that

$$\text{(3.3)} \qquad ACU_1 \cdots U_{\alpha-1} = 0 \qquad \text{and} \quad CDV_1 \cdots V_{\beta-1} = 0,$$

and each such term contributes q^r to the sum, where $r = ms_{\alpha-1} + nt_{\beta-1}$. So, as $U_1, \cdots, U_{\alpha-1}, V_1, \cdots, V_{\beta-1}$ vary over all matrices of their respective sizes with elements from F, we must determine how many times (3.3) is satisfied. Hence, for any fixed $C = C(n, s; z)$ we must find the number of solutions to the matric equations in (3.3). It is at this point we need the added restrictions on the matrices A and D. We first discuss the left equation in (3.3). The number of solutions, which we call $N_\alpha(z)$, of $ACU_1 \cdots U_{\alpha-1} = 0$ is a special case of [5; Theorem 1]. However, it is shown in this paper that $N_\alpha(z)$ is a function of the rank of the constant matrix AC so that we must know the rank of AC. If the rank of A is not equal to the number of columns of A then, in general, the rank of AC is not a function of only the rank of C. But, with $A = A(m, n; n)$ then we have rank AC = rank C = z. Hence,

$N_\alpha(z)$ is given by [5; Theorem 1] to be

$$(3.4)\qquad N_\alpha(z) = q^T \sum_{j_{\alpha-2}=0}^{(s_{\alpha-1},z)} g(z, s_{\alpha-1}; j_{\alpha-2})\, q^{-s_{\alpha-2} j_{\alpha-2}} \cdot$$

$$\cdot \prod_{i=1}^{\alpha-2} \sum_{j_{\alpha-i-1}=0}^{(j_{\alpha-i}, s_{\alpha-i})} g(j_{\alpha-i}, s_{\alpha-i}; j_{\alpha-i-1})\, q^W,$$

with $T = s_{\alpha-1}(s_{\alpha-2} - z) + ss_1 + \cdots + s_{\alpha-3}\, s_{\alpha-2}$ and $W = -j_{\alpha-i-1}\, s_{\alpha-i-1}$,

where (u, v) = minimum of u and v; $g(a, b; y)$ is given by (2.4); the sum over any j_k is defined to be 1 when the upper limit is zero; the product over i is defined to be 1 for $\alpha = 2$, and for $\alpha = 2,3$ $s_{\alpha-2}$, $s_{\alpha-3}$ are defined to be 0.

By using a similar discussion, we may determine the number $N_\beta(z)$ of solutions of $CDV_1 \cdots V_{\beta-1} = 0$ from (3.4) by replacing α with β and s_k with t_k in this expression. The theorem now follows by substituting the value $N_\alpha(z)\, N_\beta(z)\, q^r$ into (3.1), noting (2.5), and simplifying the resulting expression.

4. Some particular results

It is perhaps of some interest to consider (1.1) in the cases $\alpha = 1$, $\beta \geq 2$ and $\alpha \geq 1$, $\beta = 1$. Clearly, $\alpha = \beta = 1$ is given by Hodges [3]. The details of the proofs of these cases are somewhat like the proof of Theor. 1 so will not be included.

THEOREM 2. *Let* $\alpha = 1$, $\beta \geq 2$ *be integers with* $A, B, D, V_1, \cdots, V_\beta$ *as defined in Theor. 1 and* $U_1 = U_1(s, m)$. *Then the number* $N(1, \beta)$ *of partitions of* B *as defined by* (1.1) *is given by* $q^\gamma N_\beta(0)$ *where* $\gamma = s(m - n) + nt_{\beta-1}$, *and* $N_\beta(0)$ *is defined in Theor. 1.*

THEOREM 3. *Let* $\alpha \geq 1$, $\beta = 1$, $A, B, D, U_1, \cdots, U_\alpha$ *be as in Theor. 1 with* $V_1 = V_1(t, n)$. *Then the number of partitions of* B *as defined by* (1.1) *is given by* $q^\delta N_\alpha(0)$, *where* $\delta = n(t - s) + ms_{\alpha-1}$, *and* $N_\alpha(0)$ *is given by* (3.4).

5. The general partition

For each $1 \leq h \leq k$, we define $A_h = A_h(m_h, n_h; n_h)$, $D_h = D_h(s_h, t_h; s_h)$ and $A_h(U_h, V_h)\, D_h = U_{h1} \cdots U_{h\alpha_h} A_h + D_h V_{h1} \cdots V_{h\beta_h}$ where $U_{h1} = U_{h1}(s, s_{h1})$, $U_{hi} = U_{hi}(s_{h,i-1}, s_{h,i})$ for $1 < i < \alpha_h$, $U_{h\alpha} = U_{h\alpha}(s_{h,\alpha-1}, m_h)$, $V_{h1} = V_{h1}(t_h, t_{h1})$, $V_{hj} = V_{hj}(t_{h,j-1}, t_{h,j})$ for $1 < j < \beta_h$, $V_{h\beta} = V_{h\beta}(t_{h,\beta-1}, n)$. We now seek the number of ways a matrix $B = B(s, n; \rho)$ may be partitioned as

$$(5.1)\qquad A_1(U_1, V_1)\, D_1 + \cdots + A_k(U_k, V_k)\, D_k = B.$$

It is possible to prove.

THEOREM 4. *If* $\alpha_h, \beta_h \geq 2$, $1 \leq h \leq k$, *then the number* N_k *of partitions of a matrix* $B = B(s, n; \rho)$ *as described by* (5.1) *is given by*

$$N_k = q^{R-sn} \sum_{z=0}^{(n,s)} H(B, z) \prod_{h=1}^{k} N_{\alpha h}(z) N_{\beta h}(z),$$

where $R = r_1 + \cdots + r_k$ *with* $r_h = ms_{h,\alpha-1} + nt_{h,\beta-1}$, $H(B, z)$ *is defined by* (2.6), $N_{\alpha h}(z)$, $N_{\beta h}(z)$ *are defined immediately following* (5.3), *and* $(n, s) =$ *minimum of* n *and* s.

Proof. In view of (2.3), we may write

$$N_k = q^{-st} \sum_{C} \sum (U_{hi}, V_{hj}) e\{\sigma([A_1(U_1, V_1) D_1 + \cdots + A_k(U_k, V_k) D_k - B]C\},$$

where the sum over C is over all $C = C(n, s)$ and $\Sigma(U_{hi}, V_{hj})$ denotes a summation over each U_{hi}, V_{hj}, $1 \leq h \leq k$, as these matrices are defined above. Now if we divide the sum over C into successive sums over all $C = C(n, s; z)$, $0 \leq z \leq (n, s) =$ minimum of n and s, and note (2.2), we may write the above line as

$$(5.2) \qquad N_k = q^{-st} \sum_{z=0}^{(n,s)} \sum_{C} e\{-\sigma(BC)\} \prod_{h=1}^{k} W_h, \qquad \text{where:}$$

$$W_h = S(U_{h1}, \cdots, U_{h\alpha_h}, V_{h1}, \cdots, V_{h\beta_h}) e\{\sigma[A_h(U_h, V_h) D_h C]\}.$$

If we make appropriate substitutions into (3.1) through (3.4), we may obtain the value of W_k to be

$$(5.3) \qquad W_h = q^{r_h} N_{\alpha h}(z) N_{\beta h}(z),$$

where $r_h = ms_{h,\alpha-1} + nt_{h,\beta-1}$, $N_{\alpha h}(z)$ is obtained from (3.4) by letting $s_a = s_{h,a}$ for all subscripts a, $\alpha = \alpha_h$; and $N_{\beta h}(z)$ is obtained from N_β by letting $t_a = t_{h,a}$, $\beta = \beta_h$. The theorem now follows by substituting (5.3) into (5.2) and noting (2.4).

We note that theorems corresponding to Theor. 4 when some or all of α_k and (or) $\beta_k = 1$ can be obtained, but we shall not dwell on that.

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