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On a generalization of Riesz operators, II

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Analisi funzionale. — *On a generalization of Riesz operators*, II. Nota di GHEORGHE CONSTANTIN, presentata [(*)] dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà nuove proprietà spettrali e di struttura per la classe $\mathscr{R}$ di operatori definiti nella Nota I.

1. A class of operators on Hilbert spaces which generalizes the class of operators with compact imaginary part is given in [2] by following

DEFINITION. *A bounded linear operator* T *on a Hilbert space* H *is said to be* $\tilde{\mathscr{R}}$ *class if every point* $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ *is a Riesz point of* $\sigma(T)$.

The aim of this Note is to give new properties about this class of operators.

2. Let $T \in \tilde{\mathscr{R}}$ and let $M \subset H$ be a closed subspace which is invariant under T. We denote by T_M the operator induced by T in the quotient space H/M defined by

$$T_M(x + M) = T_M x + M.$$

We shall need the following lemma [6]: Let $T \in \mathscr{L}(X)$ have a connected resolvent set. If M is a closed subspace of X invariant under T then M is invariant under $R(z\,;T)$ for all $z \in \rho(T)$.

We have.

PROPOSITION 2.1. 1) $\rho(T) \subseteq \rho(T|_M)$ *and* $R(z\,;T)|_M = R(z\,;T|_M)$ *for all* $z \in \rho(T)$

2) $\rho(T) \subseteq \rho(T_M)$ *and* $R(z\,;T)_M = R(z\,;T_M)$ *for all* $z \in \rho(T)$.

Proof. 1) Let $\lambda \in \rho(T)$; then $R(\lambda\,;T) \in \mathscr{L}(H)$ and $R(\lambda\,;T)|_M \in \mathscr{L}(M)$ by the above lemma since $\rho(T)$ is a connected set. Also we have

$$\{(T - \lambda I)_M\}\, R(\lambda\,;T)|_M = \{(T - \lambda I)\, R(\lambda\,;T)\}|_M = I|_M$$

and a similar result if the order of multiplication is reversed. Hence $\lambda \in \rho(T)|_M$ and $R(\lambda\,;T)|_M = R(\lambda\,;T|_M)$.

The proof of part 2) is exactly analogous.

THEOREM 2.1. *If* $T \in \tilde{\mathscr{R}}(H)$ *then*

1) $T|_M \in \tilde{\mathscr{R}}(M)$;

2) $T_M \in \tilde{\mathscr{R}}(H/M)$.

Proof. Let $\lambda \in \sigma(H/M)$, $\operatorname{Im} \lambda \neq 0$; then by Proposition 2.1 we have that $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ and

$$P(\lambda\,;T) = \int_{\sigma(\lambda)} R(\lambda\,;T)\, d\lambda$$

(*) Nella seduta del 20 aprile 1974.

where $\gamma(\lambda)$ is a closed contour contained in the resolvent set and $\sigma(T) \cap \cap \gamma(\lambda) = \{\lambda\}$. Since $P(\lambda; T)(M) \subseteq M$ we have that $P(\lambda; T)|_M \in \mathscr{L}(M)$ and M can be written

$$M = R\{P(\lambda; T)|_M\} \oplus N\{P(\lambda; T)|_M\}$$

where $R\{P(\lambda; T)\}$ is a finite dimensional subspace and thus $R\{P(\lambda; T)|_M\}$ is with this property. From the fact that $\sigma(T|_M) \subseteq \sigma(T)$ for $T \in \tilde{\mathscr{R}}(H)$, we conclude that $(T - \lambda I)|_{R\{P(\lambda;T)|_M\}}$ is nilpotent and $(T - \lambda I)|_{N\{P(\lambda;T)|_M\}}$ is a homeomorphism. Since $P(\lambda; T)|_M \neq 0$ (if $P(\lambda; T)|_M = 0$ then $\lambda \in \rho(T|_M)$) it follows that λ is a Riesz point for $\sigma(T|_M)$ and thus $T|_M \in \tilde{\mathscr{R}}(M)$.

The proof of part 2) is exactly analogous.

For every $\lambda \in \sigma(T)$, $\text{Im}\,\lambda \neq 0$ we have

$$H = N(\lambda; T) \oplus F(\lambda; T)$$

the decomposition from the definition of Riesz point, and for the points $\lambda \in \sigma_p(T)$, $\text{Im}\,\lambda = 0$, if we denote by

$$J_\lambda = \{x : (T - \lambda I)^k x = 0 \quad \text{for some integer } k \geq 1\}$$

then we have

PROPOSITION 2.2. *If* $T \in \tilde{\mathscr{R}}(H)$, $\lambda_0 \in \sigma_p(T)$ *with* $\text{Im}\,\lambda_0 = 0$ *then* $\overline{J_{\lambda_0}} \subseteq \subseteq F(\mu; T)$ *for every* $\mu \in \sigma(T)$, $\text{Im}\,\mu \neq 0$ *where* μ *is not in a circle* γ *which contains a spectral set* σ *with* $\lambda_0 \in \sigma$.

The following theorem is a generalization of some results which are given in [5], [1], [6].

THEOREM 2.2. *Let* $T \in \mathscr{L}(H)$ *and* $f(\lambda)$ *be an analytic function in a region which contains* $\sigma(T)$. *If* $\lambda_0 \in \sigma(T)$, *Im* $f(\lambda_0) \neq 0$ *and* $f(T) \in \tilde{\mathscr{R}}(H)$ *then* λ_0 *is a pole for* $R(\lambda; T)$. *If* $P(\lambda_0; T)$ *is the projection associated with the spectral set* $\{\lambda_0\}$ *then* $R\{P(\lambda_0; T)\}$ *is finite dimensional and the eigenspace corresponding to the eigenvalue* λ_0 *is also finite dimensional.*

Proof. Let $A = f(T)$ and $\mu = f(\lambda_0)$ Then it is known that $f(\lambda_0) \in \in \sigma(f(T))$ and since $f(T) \in \tilde{\mathscr{R}}(H)$ it follows that μ is an isolated point of $\sigma(f(T))$. It is also known (see [5, p. 304]) that $\sigma = \{\lambda : \lambda \in \sigma(T), f(\lambda) = f(\lambda_0)\}$ is a spectral set for T and $P_\sigma = P(\mu; A)$ where P_σ is the spectral projection associated with σ and T, and $P(\mu; A)$ that associated with μ and $f(T)$. If A_0 is the restriction of A to $R\{P_\sigma\}$, then $\sigma(A_0) = \{\mu\}$ and therefore $0 \in \rho(A_0)$. From Theorem 2.1 we conclude that $A_0 \in \tilde{\mathscr{R}}$ and since $\sigma(A_0) = \{\mu\}$, $\mu \neq 0$, it follows that A_0 is an invertible Riesz operator which implies that $R\{P_\sigma\}$ is finite dimensional. If T_1 is the restriction of T to $R\{P_\sigma\}$ then $\sigma(T_1) = \sigma$ and since $R\{P_\sigma\}$ is finite dimensional, σ is a finite set. From the fact that $\lambda_0 \in \sigma$ it follows that λ_0 is an isolated point in $\sigma(T)$. Also, if $P(\lambda_0; T)$ is the projection associated with $\{\lambda_0\}$ and T and $P(\sigma_0; T)$ that associated with $\sigma - \lambda_0$ and T, then:

$$P_\sigma = P(\sigma_0; T) + P(\lambda_0; T)$$

where $P(\sigma_0; T) P(\lambda_0; T) = 0$. Hence $R\{P(\lambda_0; T)\} \subseteq R\{P_\sigma\}$ so that $R\{P(\lambda_0; T)\}$ is finite dimensional. When T is restricted to $R\{P(\lambda_0; T)\}$, its resolvent must therefore have a pole at λ_0. But in this case it is known that this implies that λ_0 is a pole of $R(\lambda; T)$ on H.

On the other hand the eigenspace corresponding to the eigenvalue λ_0, $N\{T - \lambda_0 I\} \subseteq R\{P(\lambda_0; T)\}$ and therefore $\dim N\{T - \lambda_0 I\} < \infty$ and the theorem is proved.

Remark. If for every $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ we have $\operatorname{Im}(\lambda_0) \neq 0$ and $f(T) \in \tilde{\mathscr{R}}$ then $T \in \tilde{\mathscr{R}}$.

We recall that an operator T is said to be strongly (weakly) asymptotically convergent if the sequence $\{T^n\}$ converges strongly (weakly) in the space $\mathscr{L}(H)$.

If follows that if an operator T is strongly asymptotically convergent and $T \in \tilde{\mathscr{R}}$ then $\sigma(T) \subseteq \{\lambda : |\lambda| < 1\} \cup \{-1, 1\}$ and $-1 \notin \sigma_p(T)$.

Indeed, it is easy to see that if $\{T^n\}$ converges then $\|T^n\| \leq M < \infty$ for $n = 1, 2, \cdots$ and therefore

$$r_T = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{1/n} \leq 1 .$$

Since $T \in \tilde{\mathscr{R}}$ we conclude that every point $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ is an eigenvalue for T and on the other hand $\{\lambda : |\lambda| \geq 1, \lambda \neq 1\} \cap \sigma_p(T) = \varnothing$ because in the contrary case $\{T^n\}$ is not strongly convergent.

PROPOSITION 2.3. *Let* T *be a contraction operator on* H *and* $T \in \tilde{\mathscr{R}}$. *Then* T *is strongly asymptotically convergent if and only if*

$$\sigma_p(T) \cap \{\lambda : |\lambda| = 1\} \subseteq \{1\}.$$

Proof. From the fact that $T \in \tilde{\mathscr{R}}$, it follows that $\{\lambda \in \sigma(T), \operatorname{Im} \lambda \neq 0\}$ is at most a countable set and therefore $\sigma(T) \cap \{\lambda : |a| = 1\}$ is countable and the assertion follows from Proposition 2 [3].

In [4] J. T. Schwartz introduced the almost normal operators (i.e., $T^* T - TT^* =$ compact) which generalize the class of operators with compact imaginary part. Utilizing a result from [4] we give

PROPOSITION 2.4. *If* T *is a spectral operator almost normal and* $T \in \tilde{\mathscr{R}}$ *then* $T = S + N$ *where* $S \in \tilde{\mathscr{R}}$ *is scalar and* N *nilpotent.*

Proof. It is known that an operator S is scalar type if and only if $S = RAR^{-1}$ where A is a normal operator and R is invertible on H. If $\omega(A)$ is the Weyl spectrum of A and $\pi_{00}(A)$ the set of isolated eigenvalues of finite multiplicity, then

$$\omega(A) = \sigma(A) - \pi_{00}(A) \tag{1}$$

since for normal operators the Weyl theorem holds. It follows that the Weyl theorem holds for S from the similarity of S with A. Also we have

$$\omega(S) = \omega(T) \quad \text{and} \quad \omega(T) \supset \sigma(T) - \pi_{0f}(T)$$

where $\pi_{0f}(T)$ is the set of eigenvalues of finite multiplicity. Hence we conclude that $\omega(T)$ is a subset of the real line and also $\omega(S)$. Then

$$\{ \lambda \in \sigma(S) \, , \, \mathrm{Im}\, \lambda \neq 0 \} \subseteq \pi_{00}(S)$$

and since $\pi_{00}(S) = \pi_{00}(A)$ and A is normal we obtain that every $\lambda \in \pi_{00}(S)$ is a Riesz point of $\sigma(S)$ and therefore $S \in \tilde{\mathscr{R}}$.

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