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Random-proximal spaces

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Calcolo delle probabilità. — *Random-proximal spaces.* Nota di GHEORGHE CONSTANTIN e VIOREL RADU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Nella Nota sono introdotti gli spazi Random–Prossimali (RP) e si dimostra che ogni spazio metrico probabilistico e ogni spazio aleatorio è uno RP–spazio. Sono studiate certe topologie sugli RP–spazi e la loro metrizzabilità.

The aim of this Note is to introduce the so-called here random-proximal space. It is proved that every probabilistic metric space [8] and every random space [7], [9], [10] is a RP-space. Two topologies on a RP-space are studied and a theorem of metrisability of a RP-space is proved.

1. Let S be an arbitrary set. If α is a relation on the family $\mathscr{P}(S)$ of all subsets of S then we write $A\alpha B$ for $(A, B) \in \alpha$ and $A\bar{\alpha}B$ for $(A, B) \notin \alpha$. $\varnothing$ will be the empty set.

DEFINITION 1.1. [2, 6] A semi-proximity on S is a relation α on $\mathscr{P}(S)$ such that:

(P1) $\varnothing\bar{\alpha}S$;

(P2) $A\alpha B$ iff $B\alpha A$;

(P3) $A\alpha(B\cup C)$ iff $A\alpha B$ or $A\alpha C$;

(P4) $A\cap B \neq \varnothing \Rightarrow A\alpha B$.

A proximity is a semi-proximity for which:

(P5) If $A\bar{\alpha}B$ then there exists a subset E of S such that

$$A\bar{\alpha}E \quad \text{and} \quad B\bar{\alpha}\mathbf{C}E$$

where **C**E is the complementary (in S) of E.

If α satisfies

(P6) $\{p\}\,\alpha\,\{q\} \Rightarrow p = q$ for all p, q in S then it is called separated.

If G is a topological group (see [1]) then $\mathscr{N}$ will denote the neighborhood system at 0 (the neutral element of G) and $\bar{X}$ will be the closure of the subset X of G. Note that G is separated if and only if the set $\{0\}$ is closed and G is metrisable if and only if $\mathscr{N}$ has a countable base.

Let $\mathscr{F}$ be a mapping of $S \times S$ into G such that:

(RP1) $\mathscr{F}(p, q) = 0$ iff $p = q$;

(RP2) $\mathscr{F}(p, q) = \mathscr{F}(q, p)$ for every p, q in S.

If A and B are subsets of S then we write

$$\mathscr{F}(A, B) = \{\mathscr{F}(p, q), \text{ where } p \text{ is in A and } q \text{ in B}\}.$$

(*) Nella seduta del 28 maggio 1974.

DEFINITION 1.2. A random-proximal space (RP-space) is a quadruple $(S, \alpha, \mathscr{F}, G)$ such that

(RP3) $A\alpha B$ iff $0 \in \overline{\mathscr{F}(A, B)}$

where α is a semi-proximity on S.

In [7, 9, 10] are introduced the so called random spaces and it is proved that every Menger space (see [8]) is a random space. In what follows the relation between these spaces and the RP-spaces is studied.

Concerning the probabilistic metric spaces and the RP-spaces we have:

THEOREM 1.1. *Every probabilistic metric space is a* RP-*space.*

Proof. Let G be the family of all measurable functions (random variables) on a probability space $(\Omega, \mathscr{K}, P)$ with the convergence in repartition (see [7]). We can consider that the distribution functions considered in [8] correspond to elements of such a G (see also [9, 10]). Now define

$$\mathscr{F}(p, q) = \xi_{pq} \in G$$

which correspond to F_{pq}.

R. Frische has proved that the relation α on $\mathscr{P}(S)$ defined via

$A\alpha B$ iff for every positive real numbers ε and λ there exists elements p in A and q in B such that $F_{pq}(\varepsilon) > 1 - \lambda$ is a semi-proximity on S.

It is obvious that (RP1) and (RP2) are satisfied. It remains to prove (RP3).

If $0 \in \overline{\mathscr{F}(A, B)}$ then there exists a sequence $\xi_{p_n q_n}$ in G ($p_n \in A$, $q_n \in B$) such that $\xi_{p_n q_n}$ converges to 0. This implies that for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer n_0 such that if $n \geq n_0$ then $F_{p_n q_n}(\varepsilon) > 1 - \lambda$ that is $A\alpha B$.

Conversely, if $A\alpha B$ and n is a positive integer then there exists p_n in A and q_n in B such that

$$F_{p_n q_n}\left(\frac{1}{n}\right) > 1 - \frac{1}{n}.$$

Now if x and λ are positive real numbers then there exists a positive integer n_0 such that $\frac{1}{n_0} < x$ and $1 - \frac{1}{n_0} > 1 - \lambda$. If n is a positive integer greater than or equal to n_0 then

$$F_{p_n q_n}(x) \geq F_{p_n q_n}\left(\frac{1}{n}\right) > 1 - \frac{1}{n} > 1 - \lambda$$

which implies that $\xi_{p_n q_n} \to 0$ that is $0 \in \overline{\mathscr{F}(A, B)}$ and the theorem is proved.

THEOREM 1.2. *Every random space is an* RP-*space.*

Proof. In [10] it is proved that the family $\mathscr{U} = \{U_N\}_{U \in \mathscr{N}}$ where $U_N = \{(p, q) \in S \times S, \mathscr{F}(p, q) \in N\}$ is a uniformity on S. Let α be the semi-proximity on S induced by $\mathscr{U}$ [2 6] that is

$$A\alpha B \quad \text{iff} \quad U_N(A) \cap B \neq \varnothing \quad \text{for every } N \text{ in } \mathscr{N}.$$

We will prove that (RP3) is satisfied.

If $A\alpha B$ then $U_N(A)\cap B = \varnothing$ implies that there exists p in A and q in B such that $(p, q) \in U_N$ or $\mathscr{F}(p, q) \in N$. Thus for every N in $\mathscr{N}$, $\mathscr{F}(A, B)\cap \cap N \neq \varnothing$ which implies $0 \in \overline{\mathscr{F}(A, B)}$.

Conversely if $0 \in \overline{\mathscr{F}(A, B)}$ then for every N in $\mathscr{N}$, $\mathscr{F}(A, B)\cap N \neq \varnothing$ that is there exists p in A and q in B such that $\mathscr{F}(p, q) \in N$ which implies that $(p, q) \in U_N$ and thus $U_N(A)\cap B \neq \varnothing$ that is $A\alpha B$. Since the first two conditions in the definition of a random space are exactly (RP1) and (RP2) then the theorem is proved.

2. Let σ be the topology on S induced by the closure operator $A \to \{p \in S, \{p\}\, \alpha A\}$.

It is known that if α is a proximity then σ is uniformisable (see [2, 6]).

It is easy to see that if G is separated then α is separated ($\{p\}\, \alpha \, \{q\} \Longleftrightarrow \Longleftrightarrow 0 \in \overline{\mathscr{F}(p, q)} \Longleftrightarrow \mathscr{F}(p, q) = 0 \Longleftrightarrow p = q$).

THEOREM 2.1. *σ is coarser than the topology $\breve{\sigma}$ induced by the mappings*

$$f_q : S \to G \quad , \quad f_q(p) = \mathscr{F}(p, q)$$

Proof. By [1] $\breve{\sigma}$ is the coarsest topology on S such that all f_q are continuous. Also $\breve{\sigma}$ is uniformisable and it is given by the coarsest uniformity on S for which all f_q are uniformly continuous.

Now let A in σ and (p_n) a net in his complementary **C**A. Suppose that p_n converges to a point p in the topology $\breve{\sigma}$. Then $\mathscr{F}(p_n, p)$ converges to $\mathscr{F}(p, p) = 0$ in G that is $0 \in \overline{\mathscr{F}(p, \mathbf{C}A)}$ and thus $\{p\}\, \alpha\, \mathbf{C}A$ which implies p is in **C**A. Therefore (p_n) cannot converge to a point of A that is A is in $\breve{\sigma}$. i.e. $\sigma \subset \breve{\sigma}$.

Remark. If G is separated then $\breve{\sigma}$ is separated.

Proof. If $p \neq q$ then $\mathscr{F}(p, q) \neq 0$. Let N_0 and N_{pq} be two neighborhoods of 0 and $\mathscr{F}(p, q)$ in G. By the continuity of $f_p : r \to \mathscr{F}(r, p)$ et $r = p$ and $r = q$ we obtain two disjoint neigborhoods of p and q.

Let $(S, \alpha, \mathscr{F}, G)$ be an RP-space and suppose that G is metrisable. Let $N_1, N_2, \cdots$ a countable base for $\mathscr{N}$.

THEOREM 2.2. *If $\breve{\sigma}$ is compact then it is metrisable.*

Proof. Since $f_q : p \to \mathscr{F}(p, q)$ is continuous at q then there exists for every positive integer k an open neighborhood $V_k(q)$ such that $V_k(q) \subset f_q^{-1}(N_k)$. Now let q_k a convergent sequence in the topology $\breve{\sigma}$, $q_k \to q$. If $p \in \bigcap_{k=1}^{\infty} f_{q_k}^{-1}(N_k)$ then for every k, $f_{q_k}(p) \in N_k$ that is $\mathscr{F}(q_k, p)$ converges to 0 in G. But $\mathscr{F}(q_k, p)$ converges to $\mathscr{F}(q, p)$ that is $\mathscr{F}(q, p) = 0$ and thus $q = p$. Since $\bigcap_{k=1}^{\infty} V_k(q_k) \subset \subset \bigcap_{k=1}^{\infty} f_{q_k}^{-1}(N_k)$ then the intersection $\bigcap_{k=1}^{\infty} V_k(q_k)$ contains at most a point. Then by [5] $\breve{\sigma}$ is metrisable.

COROLLARY. *If* $\breve{\sigma}$ *is compact and* σ *is separated then it is compact and metrisable.*

Proof. Since $\sigma \subset \breve{\sigma}$ then by considering the identity map $i:(S, \breve{\sigma}) \rightarrow$ $\rightarrow (S, \sigma)$ then we obtain that i is a homeomorphism.

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