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On the ostensible steady state of a dynamical system

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the ostensible steady state of a dynamical system.* Nota (*) di RUTHERFORD ARIS, presentata dal Socio straniero C. TRUESDELL.

RIASSUNTO. — In questo scritto si distingue il concetto di stato stazionario ostensibile di un sistema dinamico da quello del vero stato stazionario come condizione che prevale quando *inputs* costanti producono *outputs* costanti. Si danno esempi per dimostrare che i due stati stazionari, sebbene spesso si raggiungano simultaneamente, sono distinti logicamente dacché lo stato stazionario ostensibile si raggiunge in un tempo finito mentre il vero stato stazionario può essere approssimato asintoticamente.

INTRODUCTION

The classical definition of a dynamical system is that it is a triplet $(X, R, \tilde{\omega})$ whose members are a metric space X (the state space), the set of real numbers R (the time axis) and a map $\tilde{\omega}$: $X \times R \to X : (x, t)$ satisfying the axioms:

1) $\tilde{\omega}(x, 0) = x, \quad \forall x \in X;$

2) $\tilde{\omega}(\tilde{\omega}(x, t_1), t_2) = \tilde{\omega}(x, t_1 + t_2), \quad \forall x \in X, t_1, t_2 \in R;$

3) $\tilde{\omega}$ is continuous.

Such a definition leads to a full and satisfying mathematical theory (see, for example, Bhatia and Szego, 1970), but with an eye to its more banausic applications in control theory, this definition has been amplified to make explicit the role played by the inputs to and outputs from the system (Kalman, Falb and Arbib 1969). To the formalities of this extended definition we shall turn in the next section, but for the moment let us continue in an informal vein.

(*) Pervenuta all'Accademia il 1° luglio 1974.

The invariant set of a dynamical system includes those states which are independent of time and closed sets of states, such as limit cycles, in which the system may remain indefinitely. More formally it is a subset S of X consisting of all states x such that

$$\tilde{\omega}(x, t) \in \mathrm{S}, \quad \forall t, x \in \mathrm{S}$$

within this invariant set we distinguish isolated points known as critical points, or, to speak more familiarly, steady states of the system. With the amplified definition of a dynamical system we may discern the possibility of having an ostensible steady state in which, for constant inputs, the outputs of the system remain constant. In the ostensible steady state the internal states need not be constant since constancy is only claimed for that function of them which appears as the output of the system. That this is a very practical view of the system is evident from the fact that this ostensible steady state would be what the operator of the system would regard as its steady state, since every observation he could make of it would be constant in time.

The distinction between the two steady states would be trivial (or, at least, trite) if one were always to imply and be implied by the other. Clearly the true steady state always implies that the outputs are constant and we shall show that there are many circumstances in which the existence of the ostensible steady state does entail that of the true. But we shall also give an example of a simple system in which the ostensible steady state is reached in a finite time but the true steady state is not attained until much later.

Definition

The extended definition of a dynamical system Σ makes it an septuplet $(\mathrm{T}, \mathrm{X}, \Omega, \Gamma, \tilde{\omega}, \eta)$ whose elements have the following meaning: the time set T is an ordered subset of the reals (a generalization of the R of the classical definition which allows for a system to be defined only on a discrete set of times); X is the *state set* (in the classical definition this is a metric space); U is a set of input values and Ω a class of functions $\{\omega : \mathrm{T} \rightarrow \mathrm{U}\}$ which are the *inputs*; Y is the set of outputs values and Γ a class of *outputs* $\{\gamma : \mathrm{T} \rightarrow \mathrm{Y}\}$; $\tilde{\omega}$ is the *transition function* which defines the state $x(t)$, given that $x(\tau)$ was its state at an earlier time and the input during the interval (τ, t) is $\omega(t)$—thus

$$\tilde{\omega} : \mathrm{T} \times \mathrm{T} \times \mathrm{X} \times \Omega \rightarrow \mathrm{X} \qquad \text{with} \quad x(t) = \tilde{\omega}(t ; \tau, x(\tau), \omega);$$

finally η: $\mathrm{T} \times \mathrm{X} \rightarrow \mathrm{Y}$ is a *readout map* defining the output $y(t) = \eta(t, x(t))$ in terms of the internal state. On the input space Ω the following restrictions are imposed: firstly, that it is not empty; secondly, that if ω and $\omega' \in \Omega$ and $t_1 < t_2 < t_3$, then $\omega'' = \omega$ in $(t_1, t_2]$ and ω' in $(t_2, t_3]$ is also a member of Ω. The direction of time is implied by the fact that $\tilde{\omega}$ is defined for $t \geq \tau$ but not necessarily for $\tau > t$. The transition function satisfies the first two classical axioms in the forms:

$$\tilde{\omega}(t\,;t\,,x\,,\omega) = x \qquad \forall x \in \mathrm{X}\,, \quad t \in \mathrm{T}\,, \quad \omega \in \Omega.$$

$$\tilde{\omega}(t_3\,;t_1\,,x\,,\omega) = \tilde{\omega}(t_3\,;t_2\,,\tilde{\omega}(t_2\,;t_1\,,x\,,\omega)\,\omega)\,, \quad \forall x \in \mathrm{X}\,, t_1 < t_2 < t_3 \in \mathrm{T}.$$

The special status of the inputs is embodied in the requirement that $\tilde{\omega}(t\,;\tau\,,x\,,\omega) = \tilde{\omega}(t\,;\tau\,,x\,,\omega')$ if $\omega = \omega'$ on $(\tau\,,t]$. The third classical axiom, that of the continuity of $\tilde{\omega}$, is used to define a smooth dynamical system for which $\mathrm{T} = \mathrm{R}$, X and Ω are topological spaces and $(\tau\,,x\,,\omega) \to \to \tilde{\omega}(\cdot\,;\tau\,,x\,,\omega)$ is a continuous map $\mathrm{T}\times\mathrm{X}\times\Omega \to \mathrm{C}^1(\mathrm{T}\to\mathrm{X})$.

A dynamical system is constant iff is an additive group under the usual addition of reals, Ω is closed under the shift operator $(z^s : \omega(t) \to \omega(t+s))$, $\tilde{\omega}(t\,;\tau\,,x\,,\omega) = \tilde{\omega}(t+s\,;\tau+s\,,x\,,z^s\omega)$ for all $s \in \mathrm{T}$ and the readout map η is independent of time. We can now define the steady states of a constant dynamical system with a constant input by saying that a true steady state obtains if $x(t)$ is constant and an ostensible steady state if $y(t)$ is constant.

EXAMPLES

We shall confine attention to constant systems which can be represented by differential equations. If X is a linear vector space of n dimensions, Ω one of m dimensions and Γ of r, then a smooth linear system can be represented by

$$\dot{x} = \mathrm{F}x = \mathrm{G}\omega, \tag{1}$$

$$y = \mathrm{H}x, \tag{2}$$

when F is $n\times n$ matrix, G an $n\times m$ and H an $r\times n$. A constant state $\boldsymbol{x}$ must satisfy

$$0 = \mathrm{F}\boldsymbol{x} + \mathrm{G}\omega,$$

i.e.

$$\boldsymbol{x} = -\,\mathrm{F}^{-1}\mathrm{G}\omega \qquad \text{and} \quad \boldsymbol{y} = -\,\mathrm{HF}^{-1}\mathrm{G}\omega. \tag{3}$$

Thus $z = x - \boldsymbol{x}$ satisfies

$$\dot{z} = \mathrm{F}z \tag{4}$$

and if the eigenvalues of F have negative real parts the steady state is stable. With an initial state $x_0 = x(0)$,

$$x(t) = \boldsymbol{x} + (\exp \mathrm{F}t)\,(x_0 - \boldsymbol{x})$$

$$y(t) = \boldsymbol{y} + \mathrm{H}\,(\exp \mathrm{F}t)\,(x_0 - \boldsymbol{x}). \tag{5}$$

Then if the system is completely observable (or, as some have it, constructible) both $x(t)$ and $y(t)$ only approach their steady state values asymptotically as $t \to 0$. The approaches to the true steady state and to the ostensible one are entirely similar since, if $-\lambda$ is the greatest eigenvalue of F, each component of x and y will ultimately differ from the corresponding component of $\boldsymbol{x}$ and $\boldsymbol{y}$ by a quantity proportional to $\exp - \lambda t$.

By way of contrast we may consider a system in which both steady states are reached at the same time. Let X be the space of piecewise continuous functions on $0 \leq z \leq Z$ whose values at any instant t are $x(z, t)$ and which satisfy

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial z} + x = 0 . \tag{6}$$

Then the boundary condition at $z = 0$

$$x(0, t) = \omega \tag{7}$$

specifies the input, while the output is

$$y(t) = x(Z, t). \tag{8}$$

The steady state is a function of z satisfying

$$\frac{d\boldsymbol{x}}{dz} + \boldsymbol{x} = 0$$

giving

$$\boldsymbol{x}(z) = \omega e^{-z} \qquad \text{and} \qquad \boldsymbol{y} = \omega e^{-Z}. \tag{9}$$

If we confine attention to constant initial conditions with

$$x(z, 0) = x_0 \tag{10}$$

then

$$x(z, t) = \left\{ \begin{array}{ll} \omega e^{-z} , & 0 \leq z < t \\ x_0 e^{-t}, & t < z \leq Z \end{array} \right\}, \tag{11}$$

$$y(t) = \left\{ \begin{array}{ll} x_0 e^{-t}, & t \leq Z \\ \omega e^{-z} , & Z < t \end{array} \right\}. \tag{12}$$

Thus at the instant $t = Z$, both the true steady state $\boldsymbol{x}(z)$ and the ostensible steady state $\boldsymbol{y}$ are achieved simultaneously.

However if we consider the system governed by

$$\frac{\partial x}{\partial t} + (x - 1) \frac{\partial x}{\partial z} + x = 0 \tag{13}$$

with input

$$x(0, t) = \omega, \tag{14}$$

initial state

$$x(z, 0) = x_0 , \tag{15}$$

and output

(16) $$y(t) = x(Z, t),$$

we have an example which shows the distinction between the true and ostensible steady states. First let us observe that if

(17) $$\omega > 1 , Z > \int_1^\omega \frac{u-1}{u} du = \omega - 1 - \ln \omega,$$

then there is no continuous solution to the steady state equation

(18) $$(x - 1) \frac{dx}{dz} + x = 0.$$

In fact the steady state solution consists of a continuous segment in $0 \leq x < \zeta$ wherein $x(z)$ drops from ω to x_1, according to the equations

(19) $$z = \omega - x(z) - \ln [\omega / x(z)]$$

(20) $$\zeta = \omega - x_1 - \ln (\omega / x_1).$$

At $z = \zeta$ there is a discontinuity from $x_1 = x(\zeta - 0)$ to $x_2 = x(\zeta + 0)$ where

(21) $$\frac{1}{2}(x_1 + x_2) = 1$$

and the solution then rises continuously in $\zeta < z \leq Z$ according to the equations

(22) $$z = \zeta + x_2 - x(z) + \ln [x(z)/x_2]$$

(23) $$Z = \zeta + x_2 - 1 + \ln (1/x_2).$$

It may be shown that for any ω and Z equations (20), (21) and (23) suffice to determine ζ, x_1 and x_2. Under the assumptions of equation (17) the output is independent of ω and Z being

(24) $$y = 1.$$

We shall now show that the ostensible steady state is established in a finite time

(25) $$T_0 = \begin{cases} \ln x_0 , & x_0 > 1 , \\ 0 , & x_0 \leq 1 . \end{cases}$$

This depends only on the initial state which we take to be a constant. (Similar results can be obtained for arbitrary initial conditions, but we are concerned to give the simplest example rather than the most general result). By contrast we shall see that the true steady state is only approached asymptotically as $t \to \infty$ though there is a particular combination ω, Z and x_0, that allows it to be reached in a finite time

(26) $$T = \begin{cases} \ln (x_0/x_2) , & x_0 > 1, \\ \ln (1/x_2) \text{ or } \infty , & x_0 \leq 1. \end{cases}$$

The solution can be constructed from characteristic curves which satisfy the characteristic equations

$$\frac{dt}{ds} = 1 \quad , \quad \frac{dz}{ds} = x - 1 \quad , \quad \frac{dx}{ds} = -x . \tag{27}$$

Hence a characteristic through t', z', x' is given parametrically by

$$t = t' + s \quad , \quad z = z' + x'(1 - e^{-s}) - s \quad , \quad x = x' e^{-s}. \tag{28}$$

A family of characteristics for $t' = z' = 0$ and various x' is shown in fig. 1; the value of x decreases monotonically along a characteristic and its slope becomes negative when x drops below 1.

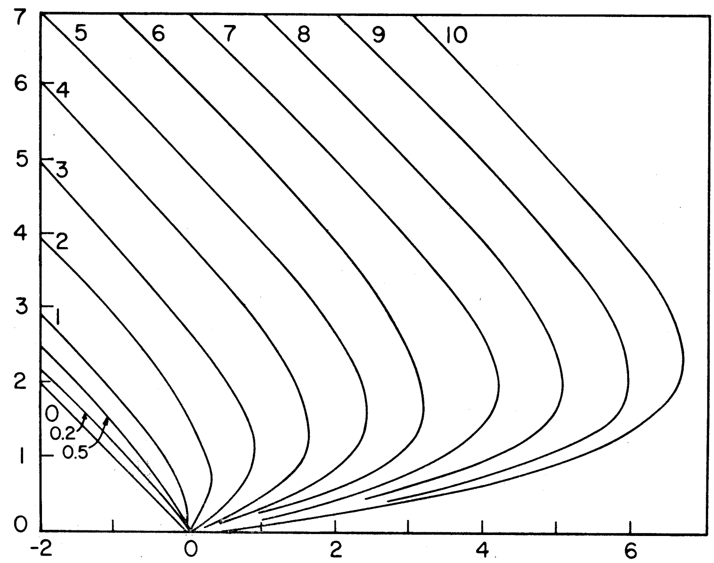

Fig. 1.

Consider first $\omega > x_0 \geq 1$ with Z satisfying the inequality (17). Then the characteristics emanating from the t-axis are all parallel and have a smaller slope from the characteristics emanating from the z-axis. In the neighbourhood of the origin two sets of characteristics would overlap did a discontinuity not intervene. At such a point as P in fig. 2. There is a discontinuity from x_1, on the left, to x_2, on the right, where x_1 is determined by following the variation of x along the characteristic AP and x_2 by following BP. Thus if s_1 and s_2 are the parameters at P along AP and BP respectively and P is $z = \xi$, $t = \tau$ we have

$$x_1 = \omega e^{-s_1} \quad , \quad \xi = \omega(1 - e^{-s_1}) - s_1 = \omega - x_1 - \ln(\omega/x_1) \tag{29}$$

and

$$x_2 = x_0 e^{-s_2} \quad , \quad \tau = s_2 = \ln(x_0/x_2). \tag{30}$$

Moreover the speed of the discontinuity is

$$\frac{d\xi}{d\tau} = \frac{1}{2}(x_1 + x_2) - 1. \tag{31}$$

The path of the discontinuity can be found by combining equations (29)-(31) to give a differential equation for x_1 in terms of x_2, namely

$$\frac{dx_1}{dx_2} = \frac{x_1(\bar{x} - 1)}{x_2(x_1 - 1)} \quad , \quad \bar{x} = \frac{1}{2}(x_1 + x_2). \tag{32}$$

This equation is integrated until

$$(x_2 - \ln x_2) - (x_1 - \ln x_1) = Z - (\omega - \ln \omega) + 1 \tag{33}$$

when a point (R, in fig. 2) is reached on a characteristic (CQR) which just touches $z = Z$ (at Q). Thereafter x_1 and x_2 must satisfy equation (33) since at a point s on the discontinuity x_1 is given by the characteristic US and x_2 by a characteristic SV which is just tangent to $z = Z$ at V. It requires a special relation between Z, ω and x_0 for (x_1, x_2) to be equal to $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ at R.

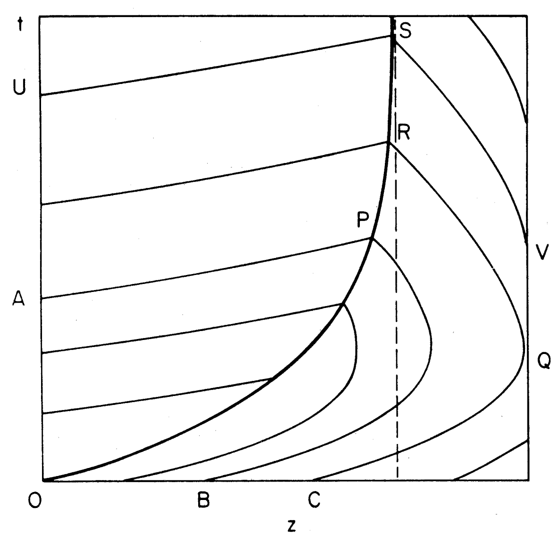

Fig. 2.

If this relation happens to be satisfied then the discontinuity has arrived at $z = \zeta$ at time $t = T = \ln(x_0/\boldsymbol{x}_2)$. In all other cases the value of $\xi \to \zeta$ as $\tau \to \infty$ and $(x_1, x_2) \to (\boldsymbol{x}_1, \boldsymbol{x}_2)$ on the curve given by equation (33). However $y(t) = x(Z, t)$ is constant and equal to 1 at all points above Q. Thus the ostensible steady state is achieved in time $T_0 = \ln x_0$ even though apart from exceptional circumstances, the true steady state is only approached asymptotically.

If $\omega > 1 > x_0 \geq 0$ we again require a discontinuity to avoid the overlapping of characteristics for the characteristics emanating from the z-axis have negative slope (cfr. fig. 3). The shock path may be traced using the same equations as before up to the point L on the characteristic DL through $z = Z$ with $x = x_0 < 1$ at D. Between DL and DM there is a centered simple wave region covered by characteristics emanating from D with values of x at D ran-

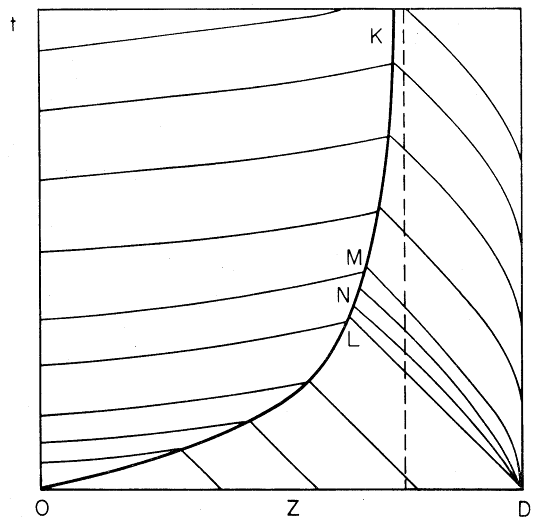

Fig. 3.

ging from x_0 to 1. Thus a point N on the shock path IM would have to satisfy

$$(34)\qquad \begin{aligned} x_1 &= \omega e^{-s_1} \quad , \quad \xi = \omega - x_1 - \ln(\omega/x_1) \\ x_2 &= x' e^{-s_2} \quad , \quad Z = \xi + x_2 - x' + \ln(x'/x_2) \quad , \quad \tau = \ln(x'/x_2). \end{aligned}$$

These equations can be combined to give an equation for x' as a function of x_1 and x_2 namely

$$(35)\qquad Z = \omega - x_1 + x_2 - x' + \ln(x' x_1/\omega x_2).$$

With x' so determined the relation between x_1 and x_2 on the shock path is given by

$$(36)\qquad \frac{dx_1}{dx_2} = \frac{x_1(x_2 - x')(\varkappa - 1)}{x_2(\varkappa - x')(x_1 - 1)} \, .$$

It is in fact convenient to regard x' as the parameter along this part of the shock path and integrate the equations

$$(37)\qquad \frac{dx_1}{dx'} = \frac{x_1(x_2 - x')(\varkappa - 1)}{x'(\varkappa - x_2)(x_1 - 1)} \quad , \quad \frac{dx_2}{dx'} = \frac{x_2(\varkappa - x')}{x'(\varkappa - x_2)}$$

$$(38)\qquad \frac{d\xi}{dx'} = -\frac{(x_2 - x')(\varkappa - 1)}{x'(\varkappa - x_2)} \quad , \quad \frac{d\tau}{dx'} = -\frac{x_2 - x'}{x'(\varkappa - x_2)} \, .$$

The first pair of equations must be integrated simultaneously but the second pair can then be integrated by quadratures. This integration continues until

$x' = 1$ when the point M is reached. Beyond this point (on MK) the path of the discontinuity approaches its steady state value asymptotically in just the same fashion as before. There will again be a particular condition on ω, Z and x_0 that makes $x_1 + x_2 = 2$ at M; in this case $x_1 = \boldsymbol{x}_1$, $x_2 = \boldsymbol{x}_2$ at M and the true steady state is achieved in finity time $T = \ln(1/\boldsymbol{x}_2)$. However we note that $y(t) = x(Z, t) = 1$ for all $t > 0$. Thus the ostensible steady state is attained instantaneously even though the true steady state is (exceptional circumstances apart) only approached asymptotically.

CONCLUSION

The examples given show that the ostensible steady state of a dynamic system is clearly distinguishable from the true steady state and, being an emminently practical concept, is worthy of being so distinguished. It has been shown that in many cases both are achieved simultaneously or approached asymptotically in the same manner, but that there are cases in which the ostensible steady state is attained in a finite time (or even instantaneously) where the true steady state is achieved at a later time or approached asymptotically.

Control theory envisiges the dynamical systems of an "input-output" character (see e.g. Kalman, Falb and Arbib 1969, p. 10) in which no attention is paid to the internal state space X. (Such are the popular boxes, of which Norbert Wiener remarked that he supposed they were black, ex officio). The ostensible steady state is thus the steady state of an input-output system. But a warning note should be added when the control of a system is studied merely in an input-output sense. Namely that there are often constraints on a real system that are as important to the internal variables as to the output and a mode of control which observes these constraints on the output but allows then to be violated within may be as fatal to the system itself as a control which disregards the constraint altogether.

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