Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.5, p. 308–310.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_5_308_0>

Matematica. — *On a* $sl(2, R)$*-module*. Nota (*) di SERGEY J. PRICHTCHEPIONOK, presentata dal Socio Straniero S. L. SOBOLEV.

RIASSUNTO. — Questa Nota è dedicata allo studio della rappresentazione dell'algebra $sl(2, R)$ in uno spazio Lorentz–invariante delle distribuzioni. Viene provato che tale rappresentazione è operatorialmente irriducibile, e viene investigata la struttura dello spazio della rappresentazione quale modulo finito SO $(2, R)$.

0. Let $O(p, q)$ be the group of all non-degenerate linear transformations of the real vector space R^n, $p + q = n$ preserving the bilinear form,

$$B_{p,q}(x\,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_{ij} y_i .$$

We denote by $O_1(p, q)$ the component of the identity of this group. By $L(p)$ and by $L_1(p)$ we mean the space of distributions defined in R^n, invariant under the action $f \to Lgf$, $g \in J$, $f \in L(p)$ $(f \in L_1(p))$ of the groups $J = O(p, q)$ and $J = O_1(p, q)$ respectively. It is known [2] that $L_1(p) = L(p)$ for $n - 1 > p > 1$ and $L_1(p) = L(p) \oplus L_n(p)$, where $L(p)$, $p = 1$ is the space of all even and $L_n(1)$ is the space of all odd Lorentz-invariant distributions.

The paper studies the algebra $sl(2, R)$ of operators in $L_1(p)$ generated by the transformations

$$\Box f = \left(\sum_{1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{p+1}^{n} \frac{\partial^2}{\partial x_i^2} \right) f \quad ; \quad Pf = B_{p,q}(x, x) f \; ;$$

$$Sf = (\Box\, P - P\, \Box) f \quad ; \quad f \in L_1(p) .$$

1. In this section we shall prove the following

THEOREM 1. *The only linear continuous intertwinning operator in* $L_n(p)$, $p = 1, n - 1$ *or in* $L(p)$ *is constant.*

Immediately obtained

COROLLARY. $L(p)$, $L_n(1)$ *are the topologically decomposable* $sl(2, R)$ — *modules.*

To prove the theorem, we use the model of the spaces $L(p)$, proposed by L. Garding and A. Tengstrand [2]. We repeat in a few words the structure of this model. Let us denote by $H_{m,\gamma}$ the space of functions defined in R of the type $\varphi_1 + \tau^m \gamma \varphi_2$, with $\varphi_i(\tau) \in C_0^\infty(R)$ $i = 1, 2$; $m = \left[\frac{n-2}{2}\right]$; $\gamma(\tau) = \ln|\tau|^{-1}$ if p, q are odd, $\gamma = \theta(\tau)$ — the Heaviside function — if p, q are even and $\gamma = \tau^{1/2}\theta(\tau)$ in other cases. Equipped with the suitable topo-

(*) Pervenuta all'Accademia il 15 ottobre 1974.

logy, $H_{m,\gamma}$ is a locally convex, complete reflexive topological space. It is dual, equipped with the Mackey, topology is topologically isomorphic to the space L (p). Using the representation and the intertwining operators continuity, we reduce our problem to the analogous problem in the space $H_{m,\gamma}$. In $H_{m,\gamma}$ we have

$$\Box = 2\left(x \frac{d^2}{dx^2} - \varkappa \frac{d}{dx}\right) \; ; \; p = x \; ; \; S = 4\left(x \frac{d}{dx} - \frac{\varkappa}{2}\right) ; \varkappa = (m - 1) \text{ or } \varkappa = m - \frac{1}{2}$$

in different cases.

Since the intertwining operator A commutes with these transformations, it may be shown (we omit all technical details) that for any $\varphi \in C_0^\infty(R)$, $A\varphi \in C_0^\infty(R)$, and $(A\varphi)' = A\varphi'$.

Using a theorem proved by J. Peetre ([3], Theorem 2) we derive that the restriction of A to C_0^∞ is a differential operator of locally finite degree. So, it is easy to see that this restriction is constant. To complete the proof, we need only to apply these arguments to the mapping $A \circ \psi$, $\psi(f) = x^m \gamma f$, $A \circ \psi : C_0^\infty(R) \to H_{m,\gamma}$.

Notice that we use the continuity of the intertwining operator only to reduce our problem to the problem in the space $H_{m,\gamma}$. So, there are no linear intertwining operators in $H_{m,\gamma}$ except constants.

In similar way prove: if $\gamma_1 \neq \gamma_2$, there are no continuous intertwining operators between H'_{m_1,γ_1} and H'_{m_2,γ_2} except the trivial ones.

2. To formulate the result of this section, we recall the definition of SO (2 , R)-finite module over *sl* (2 , R) (see for example [1]).

A *sl* (2 , R)-module is said to be SO (2 , R)-finite if it contains a dense submodule E_0 which is the straight sum of irreducible (one-dimensional) SO (2 ,R)-modules and each of these irreducible components is taken with finite order.

Let X , A , B-be the basis for *sl* (2 , R) with the conditions [X , A] = = A ; [X , B] = — B ; [A , B] = — 2 X, where $\frac{1}{i} R \cdot (X)$ is the Lie algebra of SO (2 , R). If $E_0 = \sum_{\alpha \in Z} \oplus R \cdot x_\alpha$, $X x_\alpha = \alpha x_\alpha$ denote by E_{0-} the set $\{x \in E_0, Bx = 0\}$ and $E_{0+} = \{x \in E_0, Ax = 0\}$.

It is convenient to write the topological isomorphisms between spaces L (p) , L_n (p) and $H'_{m,\gamma}$ in this form:

	n is even			n is odd	
	P is even	P is odd	$p=1$, $p=n-1$	P is even	$p=1$, $p=n-1$
L (p)	H'_{m,γ_2}	H'_{m,γ_1}	H'_{m,γ_1}	H'_{m,γ_3}	H'_{m,γ_3}
L_n (p)			H'_{m,γ_3}		H'_{m,γ_2}

Here $\gamma_1 = \ln |x|^{-1}$, $\gamma_2 = \theta(x)$; $\gamma_3 = x^{1/2}\theta(x)$. Since the structure of these spaces as the SO (2 , R)-finite modules depends on n only, denote by E_n any of these spaces.

Now we can formulate the following

THEOREM 2. 1) *E_n is the* SO (2, R)*-finite module over* $sl(2, R)$.

2) $E_0 = E_{01} \oplus E_{02} \oplus E_{03}$;

3) *If $n/4$ is not integer, then* E_{01} *is isomorphic to* E_{02}, $E_{0i} = \bigoplus_{j=-\infty}^{\infty} R \cdot X_{ij}$; $E_{0i+} = E_{0i-} = \varnothing$;

4) *If $n/4$ is integer, then* $E_{0i} = \sum_{j=-\infty}^{\infty}{}_{\oplus} R \cdot x_{ij}$; $E_{01+} = \varnothing$, $E_{01-} = Rx_{1-(n/4),1} \oplus \oplus Rx_{n/4,1}$; $E_{02-} = \varnothing$; $E_{02+} = R \cdot x_{(n/4)-1,2} \oplus R \cdot x_{-n/4,2}$;

5) *If n is odd,* $E_{03} = \varnothing$. *If n is even,* E_{03} *is the only irreducible* $(n/2 - 1)$*-dimensional* $sl(2, R)$*-module.* We omit the proof of this theorem.

By using simple arguments we find easily all such algebraic SO (2, R)-finite modules over $sl(2, R)$. Then we can see that the "chains" described in Theorem 2 are not all possible modules of this type (i.e. when the order of each eigenvalue of X is one).

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