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Oscillation of a Class of Self-Adjoint Differential Equations

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Equazioni differenziali ordinarie. — *Oscillation of a Class of Self-Adjoint Differential Equations.* Nota (*) di STEVE C. TEFTELLER, presentata dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota riguarda una classe di equazioni differenziali autoaggiunte di ordine $n \geq 3$ le cui soluzioni possono esprimersi come prodotto di soluzioni dell'equazione generale autoaggiunta $(\gamma(x) y')' + q(x) y = 0$.

1. INTRODUCTION

In [7], A. Zettl introduced the n-th order quasi differential equation

$$(E_n) \qquad (D_{n-1} y)' + \sum_{i=1}^{n-1} F_{ni} D_{i-1} y = 0,$$

where the quasi derivatives are given by

$$D_0 y = y \quad ; \quad D_1 y = F_{12}^{-1} y';$$

$$D_i y = F_{i,i+1}^{-1} \cdot \left[(D_{i-1} y)' + \sum_{j=1}^{i-1} F_{ij} y \right];$$

$i = 1, 2, \cdots, n-1$. The functions $F_{ij}(x)$; $i, j = 1, 2, \cdots, n$, are assumed to be continuous on the half axis $[\alpha, \infty)$, $F_{ij}(x) \equiv 0$ if $i + j$ is even or $j > i + 1$, $F_{i,i+1}(x) > 0$ on $[\alpha, \infty)$, for $i = 1, 2, \cdots, n-1$. J. H. Barrett established this equation for $n = 3$ and $n = 4$.

In this work, the above equation is specialized so that the equation is self-adjoint and the solutions of this "new" equation will be expressible as products of the general second order equation

$$(1) \qquad (\gamma(x) y')' + q(x) y = 0.$$

The coefficients $\gamma(x)$ and $q(x)$ are assumed to be continuous, real-valued functions on the half-axis $[\alpha, \infty)$, with $\gamma(x) > 0$ on $[\alpha, \infty)$.

P. Appel [1] showed that the product of two solutions of (1) was a solution of the third order equation $y''' + 2q(x) y' + q'(x) y = 0$, and J. H. Barrett [2, 3], extended this result to his canonical third order equation

$$(2) \qquad (\gamma(x) [(\gamma(x) y')' + 2q(x) y])' + 2q(x) \gamma(x) y' = 0.$$

(*) Pervenuta all'Accademia il 21 agosto 1974.

It is easily seen that if $\{u, v\}$ is a solution basis of (1), then the general solution of (2) is given by $y(x) = k_1 u^2(x) + k_2 u(x) v(x) + k_3 v^2(x)$, where k_1, k_2 and k_3 are constants. Hence it is immediately known that (2) is oscillatory (i.e., has a solution with zeros for arbitrarily large x) if (1) is oscillatory. More recently, the Author [5], has shown that (2) is oscillatory if and only if (1) is oscillatory.

Further, G. D. Jones [4] and W. R. Utz [6] have shown that if $y''' + 2qy' + q'y = 0$ is oscillatory, then this equation has solution bases consisting of both oscillatory and nonoscillatory solutions. In addition, the Author [5] has shown that there exists a solution basis of (2) consisting of solutions having only zeros of multiplicity n, $n = 0, 1, 2$. (A solution y has a zero of multiplicity p at b provided $D_i y(b) = 0$, $i = 0, 1, \cdots, p-1$).

The purpose of this paper is to generalize the above results to a class of self-adjoint equations of order $n > 3$.

2. A CANONICAL FORM

Let n be any integer, $n \geq 2$. Consider the n-th order quasi differential equation

$$(3) \qquad L_n(y) \equiv (D_{n,n-1}\, y)' + a_{nn}\, q(x)\, D_{n,n-2}\, y = 0,$$

where $D_{n0}\, y = y$;

$$D_{nk}\, y = \gamma(x)\, [(D_{n,k-1}\, y)' + a_{nk}\, q(x)\, D_{n,k-2}\, y];$$

$k = 1, 2, \cdots, n-1$. The constants a_{nk} are defined

$$a_{nk} = (k-1)(n-k+1), \qquad k = 1, 2, \cdots, n.$$

Considering equation (E_n), define

$$F_{ij}(x) = \begin{cases} 1/\gamma(x), & \text{if } j = i+1; \quad i = 1, 2, \cdots, n-1 \\ a_{ni}\, q(x), & \text{if } j = i-1; \quad i = 2, 3, \cdots, n. \\ 0, & \text{otherwise.} \end{cases}$$

Then equations (E_n) and (3) coincide. Furthermore,

$$F^{-1}_{i,i+1} = F^{-1}_{n-i,n-i+1} \quad ; \quad i = 1, 2, \cdots, n-1, \quad \text{and}$$

$$F_{i,i-1} = F_{n-i+2,n-i+1}\,; \quad i = 2, 3, \cdots, n,$$

which imply that (3) is self-adjoint.

It also can be seen that the constants a_{nk} can be defined by the relations $a_{n1} = 0$,

$$a_{n2} = a_{nn} = n-1 \quad , \quad a_{nk} = (n-1) + a_{n-2\,,\,k-1}\,; \quad k = 3, 4, \cdots, n-1.$$

Hence the coefficients for the n-th order equation are obtained from the coefficients of the equation of order $n - 2$. This becomes especially clear when equation (3) is written in vector-matrix form.

Finally, the quasi derivatives are doubly subscripted to avoid confusion. It is not true that $D_{mk} y = D_{nk} y$ if $m \neq n$.

3. Main results

The previously mentioned work will now be generalized.

THEOREM 1. *Suppose $n \geq 2$ is any integer. The n-th order quasi differential equation* (3) *is oscillatory if and only if equation* (1) *is oscillatory.*

Proof. Suppose $\{u, v\}$ is a solution basis for (1). It can be verified by induction that a solution basis for (3) is $\{u^m v^{n-m-1} | m = 0, 1, 2, \cdots, n-1\}$. Hence it follows immediately that (3) is oscillatory if (1) is oscillatory

$$\text{Let } w = \sum_{i=0}^{n-1} k_i u^i v^{n-i-1}$$

be a solution of (3) where the k_i*'s* are arbitrary constants.

Since w is a polynomial in u and v with real coefficients, it can be decomposed into a product of linear terms and quadratic terms in u and v, i.e., w can be written as a product of solutions of (1) and (2). Hence if $w(x)$ is an oscillatory solution of (3), then either (1) is oscillatory or (2) is oscillatory. By previous work, if (2) is oscillatory, then (1) is also oscillatory.

COROLLARY. *Suppose for equation* (3), *then $n = 2m$, $m = 1, 2, \cdots$ If* (3) *has one oscillatory solution, then every solution is oscillatory.*

Proof. The general solution of (3) can be written as $y = \sum_{i=0}^{2m-1} k_i u^i v^{2m-i-1}$ where $\{u, v\}$ is a solution basis for (1). Since y is a polynomial in u and v of odd order, it has at least one linear factor. If y is oscillatory, then this factor is an oscillatory solution of (1). Therefore, every solution of (1), and consequently (3), is oscillatory.

The following extends the results of Jones and Utz concerning the oscillation of linear combinations of (3).

THEOREM 2. *Suppose $n = 2m + 1$, $m = 1, 2, 3, \cdots$. Then if* (3) *is oscillatory, its solution space has bases consisting of $0, 1, 2, \cdots, n$ oscillatory elements.*

Proof. As was seen in Theorem 1, if y is a solution of (3), then y can be expressed as a polynomial of order $2m$ in u and v, where $\{u, v\}$ is a solution, basis of (1). Consequently, y is nonoscillatory if and only it if is the product of m irreducible quadratics in u and v.

Considering only simple roots, then it follows that the location of the roots of a polynomial depends continuously on the coefficients. Hence, by a " perturbation " argument, $2m+1$ solutions of (3) may be constructed so that each quadratic has only simple complex roots. By means of a dimensionality argument, these solutions can be chosen so that they are linearly independent.

The construction of a solution basis consisting of only oscillatory solutions requires much less finesse. If (3) is oscillatory, then (1) is oscillatory and the set $\{u^{2m}, u^{2m-1}v, \cdots, uv^{2m-1}, v^{2m}\}$ contains $2m+1$ linearly independent oscillatory solutions. A solution basis consisting of $2m+1-k$, $k=1, 2, \cdots 2m$; nonoscillatory solutions is obtained by replacing k solutions in the completely nonoscillatory basis with k solutions from the completely oscillatory basis.

A solution of (3) is said to have zeros of multiplicity k, where $k > 1$ is an odd integer, only if a zero of multiplicity k is followed by a single zero.

By considering the various factorizations of a solution of (3) into linear and quadratic factors, where such factors are solutions of (1) and (2), respectively, the following is obtained.

THEOREM 3. *Suppose* $n = 2m+1$, $m = 1, 2, 3, \cdots$. *If* (3) *is oscillatory, there exist* $2m+1$ *linearly independent solutions*, $y_0, y_1, \cdots, y_{2m}$; *where* y_k *has only zeros of multiplicity* k, $k = 0, 1, 2, \cdots 2m$.

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