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A generalized Leray-Schauder condition

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Topologia. — *A generalized Leray-Schauder condition* (*). Nota di MARIO MARTELLI e ALFONSO VIGNOLI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Sia $f: \overline{B} \to E$ una funzione continua, addensante definita nel disco unitario $\overline{B}$ di uno spazio di Banach E, e senza punti fissi sulla frontiera S di $\overline{B}$. È noto che in tal caso deg (I — f, B, 0) è definito (cfr. Nussbaum [6]) e se è diverso da zero allora il campo vettoriale $I - f: \overline{B} \to E$, $(I - f)(x) = x - f(x)$, ha almeno un punto singolare $x_0 \in B$. Una condizione che implica deg $(I - f, B, 0) \neq 0$ è la cosiddetta *condizione di Leray–Schauder*

$$\lambda x = f(x) \quad \text{per qualche} \quad x \in S \Rightarrow \lambda \leq 1 .$$

In questo lavoro si dà una condizione più generale di quella di Leray–Schauder. Essa può essere applicata anche quando f è definita sulla chiusura $\overline{\Omega}$ di un insieme aperto e limitato $\Omega \subset E$. Si rileva anche che, oltre a quella di Leray–Schauder, rientrano nella condizione qui presentata le più note *condizioni sulla frontiera* che assicurano l'esistenza di un punto singolare del campo vettoriale $I - f$.

1. Let B be the unit ball of a Banach space E and $f: \overline{B} \to E$ be a condensing map. If $x \neq f(x)$ for any $x \in \partial B$ then deg $(I - f, B, 0)$ is defined (see Nussbaum [6]) and if it is different from zero then the vector field $I - f$ vanishes at some point $x \in B$. A condition which insures that deg $(I - f, B, 0) \neq 0$ is the so-called Leray-Schauder condition:

i) $\lambda x = f(x)$ for some $x \in \partial B$ implies $\lambda \leq 1$.

Therefore if i) is verified it follows that f has a fixed point. This result can be proved also without using the machinery of the degree (see Petryshyn [7], Martelli and Vignoli [5]).

The aim of this note is to present a boundary condition which is more general than the one of Leray-Schauder. It can be applied also in the case when f is defined in the closure of a bounded open subset of E. Moreover it seems that the fixed point theorem we obtain in this way can be proved only with degree techniques.

2. Let $X \subset E$ be a subset of a Banach space E and $f: X \to E$ be a continuous map. We recall that f is said to be a condensing map (see [5]) if for any bounded and non precompact $A \subset X$ we have $\alpha(f(A)) < \alpha(A)$, where α is the Kuratowski [4] measure of noncompactness.

Let Ω be an open subset of E and denote by $\overline{\Omega}$ the closure of Ω. Let $f: \overline{\Omega} \to E$ be a condensing map. Assume that $M = \{x \in \overline{\Omega} : x - f(x) = a\}$ is compact (possibly empty). Then deg $(I - f, \Omega, a)$ is defined in the sense of Nussbaum [6].

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We are interested in the following properties of the degree.

1. (Solvability) If $\deg(I - f, \Omega, a) \neq 0$ then M is nonempty.
2. (Homotopy) Let $H : \Omega \times [0, 1] \to E$ be such that $M = \{x \in \Omega :$ there exists $t \in [0, 1]$ such that $H(x, t) = x\}$ is compact. Moreover assume that

   j) for any x_0, t_0 there exists an open neighborhood N_{x_0} and an open interval $J_{t_0} \subset [0, 1]$ such that $A \subset N_{x_0}$ and $\alpha(A) > 0$ implies
$$\alpha(H(A \times J_{t_0})) < \alpha(A)$$

Then $\deg((I - H)(\cdot, 0), \Omega, 0) = \deg((I - H)(\cdot, 1), \Omega, 0)$.

A sufficient condition which insures that j) is satisfied is the following: $H(\cdot, t)$ is a condensing map for every $t \in [0, 1]$ and $H(x, t)$ is continuous in t, uniformly in $x \in \Omega$.

3. Let Ω be a bounded open subset of a Banach space E and $g : \overline{\Omega} \to E$ be a condensing map such that $g(x) \neq x$ for every $x \in \partial\Omega$. The vector field $G : \overline{\Omega} \to E$ defined by $G(x) = x - g(x)$ is said to be solvable if $\deg(I - g, \Omega, 0) \neq 0$.

EXAMPLES OF SOLVABLE VECTOR FIELDS

1. Let Ω be a ball B about the origin and $L : E \to E$ be a condensing linear operator such that $1 \notin \sigma(L)$. Then $\deg(I - L, B, 0) \neq 0$.

2. Let Ω be as in 1. and $f : \overline{\Omega} \to E$ be a condensing map such that $f(x) = -f(-x)$, $f(x) \neq x$ for any $x \in \partial\Omega$. Then $\deg(I - f, \Omega, 0) \neq 0$.

3. Let Ω be convex and $f : \overline{\Omega} \to E$ be a condensing map such that $f(\partial\Omega) \subset \Omega$. Then $\deg(I - f, \Omega, 0) \neq 0$.

We recall that the solvability condition implies that the vector field G vanishes at some point $x \in \Omega$.

THEOREM. *Let Ω be a bounded open subset of a Banach space* E *and $f : \overline{\Omega} \to E$ be a condensing map. Assume that there exists a solvable vector field $G : \overline{\Omega} \to E$ such that the following boundary condition is verified.*

(1) $\lambda G(x) = F(x)$ *for some $x \in \partial\Omega$ implies $\lambda \geq 0$, where* $F(x) = x - f(x)$. *Then f has a fixed point.*

Proof. If for some $x \in \partial\Omega$ condition (1) is verified with $\lambda = 0$ then f has a fixed point. Therefore it is enough to prove that if (1) holds with $\lambda > 0$ then the two vector fields G and F are homotopic.

Define $H : \overline{\Omega} \times [0, 1] \to E$ by $H(x, t) = x - [tf(x) + (1 - t) g(x)]$. We prove first that H is continuous in t uniformly in $x \in \overline{\Omega}$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ and $x \in \overline{\Omega}$ implies $\| H(x, t_1) - H(x, t_2) \| < \varepsilon$.

We have

$$\| H(x, t_1) - H(x, t_2) \| \leq | t_1 - t_2 | (\| f(x) \| + \| g(x) \|)$$

Since f and g are condensing and Ω is bounded it follows that $\sup \{ \| f(x) \| + \| g(x) \| : x \in \Omega \} = N < \infty$. Therefore

$$\| H(x, t_1) - H(x, t_2) \| \leq | t_1 - t_2 | N.$$

We prove now that $H(x, t) \neq 0$ for any $x \in \partial\Omega$. Assume the contrary, i.e. $x = tf(x) + (1 - t) g(x)$ for some $x \in \partial\Omega$ and $0 \leq t \leq 1$. Hence $x - g(x) = t(f(x) - g(x))$, $t \in (0, 1)$. Thus

$$(1 - t^{-1}) G(x) = F(x), \; t^{-1} > 1.$$

This contradicts the assumption (1), therefore $H(x, t) \neq 0$ for all $x \in \partial\Omega$. It follows that H is an admissible homotopy joining the two vector fields G and F. Thus $\deg (I - f, \Omega, 0) \neq 0$ and $0 \in Im(I - f)$. Q.E.D.

Remark 1. The following conditions are clearly equivalent to condition (1).

(2) $\nu G(x) = G(x) - F(x)$ for some $x \in \partial\Omega$ implies $\nu \leq 1$.

(3) $\nu (x - g(x)) = f(x) - g(x)$ for some $x \in \partial\Omega$ implies $\nu \leq 1$.

(4) $\mu x + (1 - \mu) g(x) = f(x)$ for some $x \in \partial\Omega$ implies $\mu \leq 1$.

(5) For each $x \in \partial\Omega$ the two vectors $G(x)$ and $F(x)$ are not in opposite direction if $F(x) \neq 0$.

(6) For each $x \in \partial\Omega$ we have $tF(x) + (1 - t) G(x) \neq 0$ for any $0 \leq t < 1$

(7) For each $x \in \partial\Omega$ we have $tf(x) + (1 - t) g(x) \neq x$ for any $0 \leq t < 1$.

Remark 2. Condition (3) reduces to Leray-Schauder condition in the case when Ω is the unit ball of E and g is the zero map.

COROLLARY 1 (Granas' condition). *Let Ω be a bounded open subset of* E *and $f : \overline{\Omega} \to E$ be a condensing map. Assume that there exists a solvable vector field $G : \overline{\Omega} \to E$ such that the following boundary condition is verified.*

$$\| G(x) - F(x) \| \leq \| G(x) \|$$

for every $x \in \partial\Omega$, where $F(x) = x - f(x)$. Then f has a fixed point.

Proof. Let $\lambda G(x) = F(x)$ for some $x \in \partial\Omega$. Since $G(x) \neq 0$ for every $x \in \partial\Omega$ we have $| 1 - \lambda | \leq 1$. This clearly implies that $\lambda \geq 0$.

Remark 3. Corollary 1 was proved by Granas [2] for the case when F and G are compact vector fields.

COROLLARY 2 (Krasnosel'skij's condition). *Let Ω be the unit ball* B *around the origin and $f : \overline{B} \to E$ be a condensing map such that $x - f(x) \neq t(-x - f(-x))$ for every $x \in \partial B$ and $t > 0$. Then f has a fixed point.*

Proof. It is enough to show that there exists a solvable vector field G such that condition (1) is verified.

Put

$$g(x) = \frac{f(x) - f(-x)}{2} .$$

Clearly g is a condensing antipodal map such that $x - g(x) \neq 0$ for every $x \in \partial B$. Therefore the vector field $G : \overline{B} \to E$ defined by $G(x) = x - g(x)$ is solvable. Assume that $\lambda G(x) = F(x)$, where $F(x) = x - f(x)$. This implies that $(\lambda - 2)x - (\lambda - 2)f(x) = \lambda(-x - f(-x))$. If $\lambda = 2$ then f has a fixed point. Assume $\lambda \neq 2$. We obtain

$$x - f(x) = \frac{\lambda}{\lambda - 2}(-x - f(-x)) .$$

Therefore $\frac{\lambda}{\lambda - 2} \leq 0$, i.e. $0 \leq \lambda < 2$. Thus condition (1) is verified and the result is proved.

Remark 4. Corollary 2 was first proved by Krasnosel'skij [3] for the compact case.

COROLLARY 3. *Let Ω be a bounded open subset of a Banach space E and $f : \overline{\Omega} \to E$ be a condensing map. Assume that there exists a solvable vector field $G : \overline{\Omega} \to E$ such that one of the following boundary conditions is verified.*

i) $h(G(x)) \leq h(F(x))$, *for any* $x \in \partial\Omega$ *and one* $h \in J(G(x))$ *where* $J : E \to E^*$ *is the duality mapping* (i.e. $J(x) = \{h \in E^* : h(x) = \|x\|^2$ *and* $\|h\| = \|x\|\}$) *and* $F(x) = x - f(x)$;

ii) $h(F(x)) \leq h(G(x))$ *for any* $x \in \partial\Omega$ *and one* $h \in J(F(x))$. *Then f has a fixed point.*

Proof. i) Assume that $\lambda G(x) = F(x)$ for some $x \in \partial\Omega$. We have $h(G(x)) = \|G(x)\|^2 \leq h(\lambda G(x)) = \lambda \|G(x)\|^2$. Since $\|G(x)\| > 0$ we obtain $\lambda \geq 1$.

ii) If f has a fixed point on $\partial\Omega$ we are done. Assume that $f(x) \neq x$ for any $x \in \partial\Omega$. If $\lambda G(x) = F(x)$ for some $x \in \partial\Omega$ we have $\lambda \neq 0$ and

$$h(F(x)) = \|F(x)\|^2 \leq \frac{1}{\lambda}\|F(x)\|^2 .$$

Thus $\frac{1}{\lambda} \geq 1$, i.e. $0 < \lambda \leq 1$.

COROLLARY 4. *Let Ω be a bounded open subset of a Banach space E and $f : \overline{\Omega} \to E$ be a condensing map. Assume that there exists a solvable vector field $G : \overline{\Omega} \to E$ such that the following boundary condition is verified*

$$h(f(x)) \leq h(x)$$

for any $x \in \partial\Omega$ and one $h \in J(G(x))$, where $J : E \to E^*$ is the duality mapping. Then f has a fixed point.

Proof. Assume that $\lambda G(x) = F(x)$ for some $x \in \partial\Omega$. We have

$$0 \leq h(x - f(x)) = h(\lambda G(x)) = \lambda h(G(x)).$$

Since $G(x) \neq 0$ for any $x \in \partial\Omega$ we have $h(G(x)) = \| G(x) \|^2 > 0$. Therefore $\lambda \geq 0$.

COROLLARY 5 (Altman's condition). *Let* Ω *be a bounded open subset of a Banach space* E *and* $f : \overline{\Omega} \to E$ *be a condensing map. Assume that there exists a solvable vector field* $G : \overline{\Omega} \to E$ *such that the following boundary condition is verified*

$$\| G(x) - F(x) \|^2 \leq \| F(x) \|^2 + \| G(x) \|^2$$

for any $x \in \partial\Omega$, *where* $F(x) = x - f(x)$.

Then f *has a fixed point.*

Proof. Assume that $\lambda G(x) = F(x)$ for some $x \in \partial\Omega$. We have

$$\lambda^2 \| G(x) \|^2 \geq (1 - \lambda)^2 \| G(x) \|^2 - \| G(x) \|^2.$$

Since $G(x) \neq 0$ for any $x \in \partial\Omega$ we obtain

$$\lambda^2 \geq (1 - \lambda)^2 - 1 \qquad \text{i.e.} \quad \lambda \geq 0.$$

Remark 5. Corollary 5 contains the fixed point theorem of Altman [1] in the case when B is the unit ball of a Hilbert space H, f is compact and g is the zero map.

4. The following examples show how Theorem 1 can be used for solving some functional equations; however, this is not the only way that they can be solved.

Example 1. Let $C[0, L]$ be the Banach space of continuous real valued functions $x : [0, L] \to \mathbf{R}$ with the supremum norm and $f : \mathbf{R} \times [0, L] \to \mathbf{R}$ a continuous function such that $|f(r, t)| \leq M$ for any $(r, t) \in \mathbf{R} \times [0, L]$. Consider the operator $T : C[0, L] \to C[0, L]$ defined by

$$T(x)(t) = -\frac{tx(t)}{LM} + \int_0^t f(x(s), s)\, ds.$$

We want to show that if $M > 1$ then T has a fixed point.

The map $g : C[0, L] \to C[0, L]$ defined by $g(x)(t) = \frac{-tx(t)}{LM}$ is contractive antipodal and such that $g(x) \neq x$ for any $x \neq 0$. Let $B(0, LM)$ be the ball of radius LM centered at the origin. We have $\deg(I - g, B(0, LM), 0) \neq 0$.

Let us prove that

$$\lambda(x - g(x)) = T(x) - g(x)$$

for some $x \in \partial B(0, LM)$ implies $\lambda \leq 1$. Assume that

$$\lambda(x - g(x)) = \lambda x \left(1 + \frac{t}{LM}\right) = T(x) - g(x) = \int_0^t f(x(s), s)\, ds$$

for some $x \in \partial B(0, LM)$ and $\lambda \geq 0$. There exists $t_0 \in [0, L]$ such that $|x(t_0)| = LM$. Therefore

$$\lambda LM \left(1 + \frac{t_0}{ML}\right) = \left| \int_0^{t_0} f(x(s), s) \, ds \right| \leq \int_0^{t_0} |f(x(s), s)| \, ds \leq LM .$$

This implies that $\lambda \leq 1$. On the other hand the operator T is an α-contraction since it is the sum of a contraction with a compact map. By Theorem 1 T has a fixed point.

We remark that a fixed point of T is a solution of the differential equation

$$\begin{cases} x' \left(1 + \dfrac{t}{LM}\right) + \dfrac{x}{LM} = f(x(t), t) \\ x(0) = 0. \end{cases}$$

Example 2. Let $g : E \to E$ be a condensing map and $h : E \to E$ be a compact map. Assume that the following two conditions are verified

i) there exists $r > 0$ such that $x - g(x) \neq t(-x - g(-x))$ for any $\|x\| > r$ and $t > 0$;

ii) $$\limsup_{x \to \infty} \frac{\|h(x)\|}{\|x - g(x)\|} < 1 .$$

Then the equation $x = g(x) + h(x)$ has a solution.

Condition i) implies that if $\rho > r$ then $\deg(I - g, B(0, \rho), 0) \neq 0$. Condition ii) implies that there exist $\delta > 0$ such that $\|h(x)\| < \|x - g(x)\|$ for any $\|x\| > \delta$. Put $f(x) = g(x) + h(x)$. If R is big enough then $\lambda(x - g(x)) = f(x) - g(x)$ for some $\|x\| = R$ implies $\lambda \leq 1$. Therefore all of the conditions of Theorem 1 for the restriction of f to the closed ball $\overline{B(0, R)}$ are fulfilled. It follows that f has a fixed point.

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