Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Ada Ardito, Paolo Ricciardi, Luciano Tubaro

Dissipative Lyapunov functions and differential equations in a Banach space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 511–518.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_511_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Analisi matematica. — *Dissipative Lyapunov functions and differential equations in a Banach space.* Nota di ADA ARDITO (*), PAOLO RICCIARDI e LUCIANO TUBARO, presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si studia l'esistenza della soluzione del problema di Cauchy in spazi di Banach mediante l'introduzione di una funzione ausiliaria.

1. INTRODUCTION

Let X be a Banach space. Consider the Cauchy problem

$$\begin{cases} \dot{u} = f(t, u) \\ u(t_0) = u_0 \end{cases} \tag{1.1}$$

where $f : R \times X \to X$ is a mapping generally not continuous.

Let us suppose that there exists a mapping $V : R \times X \times X \to R$ such that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial u} f(t, u) + \frac{\partial V}{\partial v} f(t, v) \leq 0. \tag{1.2}$$

In this case we shall say that f is V-dissipative.

This concept generalizes the dissipativity when V is the norm.

If $f(t, u) = Au + g(t, u)$ where A is not a continuous linear operator, A is a semigroup generator, and $g(t, u)$ is continuous, the local existence of a solution of the problem (1.1) was proved in [7].

In this work we shall prove the existence of the solutions of the problem (1.1) when f can be approximated by continuous functions. These results generalize the works of T. Kato [4] and M. G. Crandall–A. Pazy [2] in the case of uniformly convex Banach space.

2. EXISTENCE (autonomous case)

Let X be a Banach space and let $V : R \times X \times X \to R$, $(t, x, y) \to V(t, x, y)$ be a mapping such that:

$$\begin{cases} \text{i)} \; V(t, x, y) \in C^1(R \times X \times X, R) \\ \text{ii)} \; \dfrac{\partial V(t, x, y)}{\partial t}, \quad \dfrac{\partial V(t, x, y)}{\partial x}, \quad \dfrac{V(t, x, y)}{\partial y} \\ \quad \text{are uniformly continuos} \\ \text{iii)} \; \text{there exist } \alpha, \beta, \gamma > 0 \text{ such that:} \\ \quad \alpha \|x - y\|^2 e^{-\gamma t} \leq V(t, x, y) \leq \beta \|x - y\|^2 e^{\gamma t} \; (1). \end{cases} \tag{2.1}$$

(*) Work supported by G.N.A.F.A. of C.N.R.

(**) Nella seduta del 14 dicembre 1974.

(1) If $V(t, x, y)$ does not depend by t then $\gamma = 0$.

Let $f: D_f \subset X \to X$ be a mapping $x \to f(x)$. We shall say that $f(x)$ is V–dissipative, if:

$$(2.2) \qquad \frac{\partial V(t, x, y)}{\partial t} + \frac{\partial V(t, x, y)}{\partial x} f(x) + \frac{\partial V(t, x, y)}{\partial y} f(y) \leq 0 \qquad \forall x, y \in D_f.$$

LEMMA 1. *Let $f: D_f \subset X \to X$ be a V–dissipative mapping. Consider the Cauchy problems*

$$(2.3) \qquad \begin{cases} \dot{u}_i = f(u_i) \\ u_i(0) = x_i \end{cases} \qquad i = 1, 2.$$

If $u_i: [0, \alpha[\to D_f$ $i = 1, 2$, are solutions of (2.3) *then*

$$(2.4) \qquad \| u_1(t) - u_2(t) \| \leq (\beta/\alpha)^{1/2} \| x_1 - x_2 \| e^{\gamma t/2}.$$

Proof. Let $F(t) = V(t, u_1(t), u_2(t))$. By the hypothesis that f is V–dissipative

$$(2.5) \qquad F'(t) \leq 0.$$

and then

$$(2.6) \qquad V(t, u_1(t), u_2(t)) \leq V(0, x_1, x_2).$$

By iii) in (2.1)

$$(2.7) \qquad \alpha \| u_1(t) - u_2(t) \|^2 e^{-\gamma t} \leq V(t, u_1(t), u_2(t)) \leq$$
$$\leq V(0, x_1, x_2) \leq \beta \| x_1 - x_2 \|^2$$

and the conclusion follows.

LEMMA 2. *Let $f: D_f \subset X \to X$, $x \to f(x)$, be a V–dissipative mapping. If $u: [0, \alpha[\to D_f$, $t \to u(t)$, is a solution of the Cauchy problem*

$$(2.8) \qquad \begin{cases} \dot{u} = f(u) \\ u(0) = u_0 \end{cases}$$

then

$$(2.9) \qquad \| f(u(t)) \| \leq (\beta/\alpha)^{1/2} \| f(u_0) \| e^{\gamma t/2}.$$

Proof. Let $h < \alpha - t$, $v(t) = u(t + h)$ be a solution of the Cauchy problem

$$(2.10) \qquad \begin{cases} \dot{v} = f(v) \\ v(0) = u(h). \end{cases}$$

Let $F(t) = V(t, v(t), u(t))$. Differentiate $F(t)$ and by the hypothesis of V-dissipative there results

$$(2.11) \qquad V(t, v(t), u(t)) \leq V(0, u(h), u_0)$$

and by iii) of (2.1)

$$(2.12) \qquad \| u(t + h) - u(t) \|^2 \leq \beta/\alpha \| u(h) - u_0 \|^2 e^{\gamma t}.$$

Then dividing (2.12) by h^2 and passing to the limit with $h \to 0$ the conclusion follows.

We shall say that $f : D_f \subset X \to X$ is regular (2) if there exists a sequence $\{f_n\}$, $f_n : D_f \subset X \to X$, of V-dissipative mappings such that the problem

$$(2.13) \qquad \begin{cases} \dot{u}_n = f_n(u_n) \\ u_n(0) = x \end{cases}$$

has a solution (3) for every n and in addition

$$(2.14) \qquad \begin{cases} \text{i)} \quad f_n(x) \to f(x) \qquad \forall x \in D_f \\ \text{ii)} \quad f_n = f \circ J_n \qquad \text{with} \\ \qquad \|J_n(x) - x\| = \alpha(n) \| f_n(x)\| \qquad \text{and} \\ \qquad \lim_{n\to\infty} \alpha(n) = 0. \end{cases}$$

THEOREM 3. *Let* $f : D_f \subset X \to X$, *be a regular and a* V*-dissipative mapping and let* $\{u_n(t)\}$ *be a sequence defined by* (2.13); *then* $\{u_n(t)\}$ *converges uniformly to a function* $u(t)$ *on bounded sets of* R.

Proof. By Lemma 2 we have

$$(2.15) \qquad \|f_n(u_n(t))\| \le (\beta/\alpha)^{1/2} \|f_n(x)\| e^{\gamma t/2}.$$

In addition let $F_{n,m} = V(t, u_n(t), u_m(t))$. Then

$$(2.16) \qquad F'_{n,m}(t) = \frac{\partial}{\partial t} V(t, u_n(t), u_m(t)) + \frac{\partial}{\partial x} V(t, u_n(t), u_m(t)) f_n(u_n(t)) + \frac{\partial}{\partial y} V(t, u_n(t), u_m(t)) f_m(u_m(t)).$$

From ii) in (2.14) we have $f_n(u_n(t)) = f(J_n u_n(t))$ and $f_m(u_m(t)) = f(J_m u_m(t))$ from which:

$$(2.17) \qquad F'_{n,m}(t) = \Big[\frac{\partial}{\partial t} V(t, J_n u_n, J_m u_m) + \frac{\partial}{\partial x} V(t, J_n u_n, J_m u_m) f(J_n u_n) + \frac{\partial}{\partial y} V(t, J_n u_n, J_m u_m) f(J_m u_m)\Big] + \Big[\frac{\partial}{\partial t} V(t, u_n(t), u_m(t)) - \frac{\partial}{\partial t} V(t, J_n u_n, J_m u_m)\Big] + \Big[\frac{\partial}{\partial x} V(t, u_n(t), u_m(t)) - \frac{\partial}{\partial x} V(t, J_n u_n, J_m u_m)\Big] f_n(u_n) + \Big[\frac{\partial}{\partial y} V(t, u_n(t), u_m(t)) - \frac{\partial}{\partial y} V(t, J_n u_n, J_m u_m)\Big] f_m(u_m).$$

(2) The properties that follow are fulfilled by the Yosida approximation if $-f$ is an m-accretive mapping.

(3) By Lemma 1 we have that any such solution is unique for every fixed n.

By virtue of the V-dissipativity of f the first term in the sum is ≤ 0, and by virtue of (2.15) the other terms are bounded. Thus

$$\text{(2.18)}\quad F'_{n,m}(t) \leq \left\| \frac{\partial}{\partial t} V(t, u_n(t), u_m(t)) - \frac{\partial}{\partial t} V(t, J_n u_n(t), J_m u_m(t)) \right\| + \\ + (\beta/\alpha)^{1/2} e^{\gamma t/2} \Big\{ \| f_n(x) \| \frac{\partial}{\partial x} V(t, u_n(t), u_m(t)) - \\ - \left\| \frac{\partial}{\partial x} V(t, J_n u_n(t), J_m u_m(t)) \right\| + \| f_m(x) \| \left\| \frac{\partial}{\partial y} V(t, u_n(t), u_m(t)) - \\ - \frac{\partial}{\partial y} V(t, J_n u_n(t), J_m u_m(t)) \right\| \Big\} .$$

Therefore, by iii) in (2.14) we have:

$$\text{(2.19)}\qquad \| u_n - J_n u_n \| \leq \alpha(n) \| f_n(u_n) \| \leq (\beta/\alpha)^{1/2} e^{\gamma t/2} \alpha(n) \| f_n(x) \| .$$

By the uniform continuity of $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$, there exists for every bounded set a constant $\beta_{n,m}$ such that $\lim_{n,m} \beta_{n,m} = 0$ and

$$\text{(2.20)}\qquad F'_{n,m}(t) \leq \beta_{n,m}$$

from which

$$\text{(2.21)}\qquad F_{n,m}(t) \leq \beta_{n,m}\, t$$

and therefore, we have the conclusion.

THEOREM 4. *Suppose the hypothesis of Theorem 3. If in addition for each* $\{x_n\} \in D_f$ *such that*

$$\text{(2.22)}\qquad \begin{cases} \lim_{n \to \infty} x_n = x \\ \| f(x_n) \| \leq M \end{cases}$$

then it results that

(2.23) $x \in D_f$ *and there exists* $\{x_{n_k}\} \subset \{x_n\}$ *such that* $f(x_{n_k}) \xrightarrow[\tau]{} f(x)$ *where* τ *denotes a topology on* X *weaker than the topology on* X.

Then, $u(t)$ *is differentiable in the topology* τ *on* X, $u(t) \in D_f$ $\forall t$, *and*

$$\text{(2.24)}\qquad \begin{cases} \dot{u}(t) = f(u(t)) \\ u(0) = x . \end{cases}$$

In addition the solution is unique.

Proof. Fix t, we set $x_n = J_n u_n(t)$ and $x = u(t)$. Then

$$\text{(2.25)}\qquad f(x_n) = f(J_n u_n(t)) = f_n(u_n(t))$$

and therefore

$$\text{(2.26)}\qquad \| f(x_n) \| \leq (\beta/\alpha)^{1/2} \| f_n(x) \| e^{\gamma t/2} .$$

There exists therefore a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to x such that

$$f(x_{n_k}) \to f(u(t)). \tag{2.27}$$

The conclusion follows.

3. EXISTENCE (non autonomous case)

Let $V(t, x, y)$ be a mapping $V: R \times X \times X \to R$ that satisfies the hypothesis i), ii), iii) of (2.1) and such that:

$$\left\| \frac{\partial}{\partial x} V(t, x, y) \right\| \leq \varphi(t) \|x - y\| \tag{3.1}$$

where $\varphi: R \to R$ is a mapping $t \to \varphi(t)$.

We shall say that $f: D_f \subset R \times X \to X$, $(t, x) \to f(t, x)$ is a V-dissipative mapping if:

$$\frac{\partial}{\partial t} V(t, x, y) + \frac{\partial}{\partial x} V(t, x, y) f(t, x) + \frac{\partial}{\partial y} V(t, x, y) f(t, y) \leq 0 \tag{3.2}$$

$$\forall (t, x), (t, y) \in D_f.$$

We suppose, in addition, the following condition:

$$\|f(t, u) - f(s, u)\| \leq K(u) \cdot |t - s| \tag{3.3}$$

where $K: D_f \to R$ is a mapping $u \to K(u)$.

LEMMA 1. *Let $f: D_f \subset R \times X \to X$ a V-dissipative mapping. We consider the Cauchy problems*

$$\begin{cases} \dot{u}_i = f(t, u_i) \\ u(0) = x_i \end{cases} \qquad i = 1, 2. \tag{3.4}$$

If $u_i: [0, \alpha[\to X$ are solutions of (3.4) *then*

$$\|u_1(t) - u_2(t)\| \leq (\beta/\alpha)^{1/2} \|x_1 - x_2\| e^{\gamma t/2}. \tag{3.5}$$

Proof. The proof is similar to that of Lemma 1, par. 2.

LEMMA 2. *Let $f: D_f \subset R \times X \to X$ a V-dissipative mapping, satisfying condition* (3.3). *If $u: [0, \alpha[\to X$ is a solution of the Cauchy problem*

$$\begin{cases} \dot{u} = f(t, u) \\ u(0) = u_0 \end{cases} \tag{3.6}$$

then

$$\|f(t, u(t))\| \leq (\beta/\alpha)^{1/2} \|f(0, u_0)\| e^{\gamma t/2}. \tag{3.7}$$

Proof. Set $v(t) = u(t+h)$. We observe that for $h < \alpha - t$, $v(t)$ is a solution of the Cauchy problem

$$(3.8) \qquad \begin{cases} \dot{v} = f(t+h, v) \\ v(0) = u(h). \end{cases}$$

We consider $F(t) = V(t, u(t), v(t))$. Differentiating we obtain

$$(3.9) \qquad F'(t) = \frac{\partial}{\partial t} V(t, u(t), v(t)) + \frac{\partial}{\partial x} V(t, u(t), v(t)) f(t, u(t)) + \\ + \frac{\partial}{\partial y} V(t, u(t), v(t)) f(t+h, v(t)).$$

By the hypotesis that f is V-dissipative

$$(3.10) \qquad F'(t) \leq \frac{\partial}{\partial y} V(t, u(t), v(t)) \{f(t+h, v) - f(t, v)\}$$

and by the condition (3.3)

$$(3.11) \qquad F'(t) \leq \varphi(t) \| u(t+h) - u(t) \| K(v) |h|.$$

Integrating, we obtain

$$(3.12) \qquad F(t) \leq F(0) + \int_0^t \varphi(s) K(v(s)) \| u(s+h) - u(s) \| \cdot |h| \, ds$$

from which

$$(3.13) \qquad \alpha \| u(t+h) - u(t) \|^2 e^{-\gamma t} \leq \beta \| u(h) - u_0 \|^2 + \\ + \int_0^t \varphi(s) K(v(s)) \| u(s+h) - u(s) \| \cdot |h| \, ds.$$

Dividing by $|h|^2$ and taking the limit as $h \to 0$

$$(3.14) \qquad e^{-\gamma t} \| f(t, u(t)) \|^2 \leq \beta/\alpha \| f(0, u_0) \|^2 + \\ + \int_0^t \frac{\varphi(s) K(v(s))}{\alpha} \| f(s, u(s)) \| \, ds.$$

Set $w(t) = e^{-\gamma t} \| f(t, u(t)) \|^2$ and $\psi(s) = \frac{\varphi(s) K(v(s))}{\alpha} e^{\gamma s/2}$ and taking into account the inequality $2ab \leq a^2 + b^2$, we have

$$(3.15) \qquad w(t) \leq \beta/\alpha \cdot w(0) + \frac{1}{2} \int_0^t \psi(s)^2 \, ds + \frac{1}{2} \int_0^t w(s) \, ds.$$

Then, for each bounded set $[0, T[$

$$(3.16) \qquad w(t) \leq c + \frac{1}{2} \int_0^t w(s) \, ds$$

where

$$c = \beta/\alpha\, w_0 + \frac{1}{2}\int_0^T \psi(s)^2\, ds.$$

Therefore, by Gronwall's Lemma

$$w(t) \leq c e^{t/2} \tag{3.17}$$

from which the conclusion is obvious.

We shall say that $f : D_f \subset R \times X \to X$ is regular if there exists a sequence $\{J_n(t, x)\}$ of functions $J_n : R \times X \to X$ such that if we set $f_n(t, x) = f(t, J_n(t, x))$, then

(3.18)
i) $\lim_n f_n(t, x) = f(t, x) \quad \forall (t, x) \in D_f$

i) there exists $\alpha : N \to R^+$, $\lim_n \alpha(n) = 0$ such that

$$\| J_n(t, x) - x \| \leq \alpha(n) \| f_n(t, x) \|$$

iii) for each n, $f_n(t, x)$ is V-dissipative and there exists a solution $u_n(t)$ of the Cauchy problem

$$\begin{cases} \dot{u}_n = f_n(t, u_n) \\ u_n(0) = u_0 . \end{cases}$$

Observation 5. By Lemmas 1 and 2 the solutions $u_n(t)$ are unique and

$$\| f_n(t, u_n(t)) \| \leq (\beta/\alpha)^{1/2} e^{\gamma t/2} \| f_n(0, u_0) \| . \tag{3.19}$$

THEOREM 4. *Let $f : D_f \subset R \times X \to X$ be regular, V-dissipative, and satisfy condition (3.3). In addition let $\{u_n(t)\}$ be a sequence defined as in* iii) *of* (3.18). *Then, $\{u_n(t)\}$ converges uniformly on bounded sets of* R *to a function $u(t)$.*

Proof. The proof is similar to Theorem 4, par. 2.

THEOREM 5. *Suppose the hypothesis of Theorem 4. If, in addition, for each $\{x_n\} \subset X$ such that $(t, x_n) \in D_f$ and*

$$\begin{cases} \lim_n x_n = x \\ \| f(t, x_n) \| \leq M \end{cases} \tag{3.20}$$

implies

(3.21) *$(t, x) \in D_f$ and there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $f(x_{n_k}) \xrightarrow[\tau]{} f(x)$ where τ denotes a topology on* X *weaker than the norm topology on* X.

Then $u(t)$ is differentiable in the topology τ, $(t, u(t)) \in D_f$ and

$$\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(0) = x . \end{cases} \tag{3.22}$$

References

[1] H. Brezis (1973) – *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, North-Holland.

[2] M. G. Crandall and A. Pazy (1969) – *Semigroups of non linear contractions and dissipative sets*, « J. Funct. Anal. », *3*, 376–418.

[3] G. da Prato (1971–72) – *Analisi non lineare*, Seminario Università di Roma.

[4] T. Kato (1967) – *Non linear semigroups and evolution equations*, « J. Math. Soc. Japan », *19* (4), 508–520.

[5] G. E. Ladas and V. Lakshmikantham – *Differential equations in Abstract Spaces*, Academic-Press.

[6] V. Lakshmikantham and S. Leela – *Differential and Integral inequalities*, Vol. I–II, Academic Press.

[7] P. Ricciardi and L. Tubaro (1973) – *Local Existence for differential equations in Banach Space*, « Boll. U.M.I. » (4) *8*, 306–316.