Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Periodic Solutions of Certain n-th Order Nonlinear Differential Equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **57** (1974), n.6, p. 519–524.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_57_6_519_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Equazioni differenziali ordinarie. — *Periodic Solutions of Certain n–th Order Nonlinear Differential Equations.* Nota di S. H. CHANG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Applicando il Teorema del punto fisso di Schauder si dimostra l'esistenza di soluzioni periodiche di un'equazione differenziale ordinaria di ordine n quasi non lineare.

1. INTRODUCTION

Consider the following n-th order nonlinear ordinary differential equation

$$x^{(n)} + f(t, x) = p(t), \left({}^{(n)} = \frac{d^n}{dt^n} \right) \tag{1}$$

where n is an integer ≥ 2, f is continuous and $f(t + T, x) = (f(t, x)$ for all (t, x) and for some $T > 0$, p is continuous and $p(t + T) = p(t)$ for all t, and $\int_0^T p(u)\, du = 0$. It is the purpose of this paper to prove the existence of periodic solutions with period T for the equation (1).

When $f(t, x) = f(x)$ and $n = 2$, the equation (1) has been studied by Harvey [3], Lazer [4], Leach [5], Loud [6], Opial [8], and Seifert [10]. When $n = 2$ we have established in [1] the existence of T-periodic solutions for (1) by assuming

$$\lim_{|x| \to \infty} \frac{|f(t, x)|}{|x|} = 0 \tag{2}$$

uniformly in t. In a recent paper Mawhin [7] has considered quasibounded nonlinearity, a concept generalizing (2), in studying certain functional differential equations. We may define quasiboundedness directly for the function f in (1) as follows: f is said to be *quasibounded* if the number

$$|f| = \min_{0 < \rho < \infty} \left(\max_{\substack{|x| \geq \rho \\ 0 \leq t \leq T}} \frac{|f(t, x)|}{|x|} \right) \tag{3}$$

is finite; in this case, $|f|$ is called the *quasinorm* of f. In establishing the existence of periodic solutions, Mawhin [7] has essentially assumed that the nonlinearities in his equations are quasibounded (in the above sense) and have zero quasinorms. In this paper we shall prove the existence of T–periodic solutions to the equation (1) by requiring f to be quasibounded and have a quasinorm smaller than certain positive number.

(*) Nella seduta del 14 dicembre 1974.

In section 2 we prove the basic existence Theorem. We shall apply Schauder's fixed point Theorem and use a technique generalizing those used in [1] and Lazer [4]. In section 3 we discuss the existence of even and odd T–periodic solutions.

2. Existence Theorems

Let X denote the Banach space of continuous T-periodic functions with the supremum norm, i.e. for an $\varphi \in X$, $\| \varphi \| = \max_{0 \leq t \leq T} | \varphi(t) |$.

THEOREM 2.1. *Let f be continuous and $f(t+T, x) = f(t, x)$ for all (t, x) and for some $T > 0$, p continuous and $p(t+T) = (p(t)$ for all t, and $\int_0^T p(u)\, du = 0$. Assume that there is a positive number M such that for $|x| \geq M$ and for all t either $xf(t, x) \geq 0$ or $xf(t, x) \leq 0$. If the function f is quasi-bounded with a quasinorm*

$$|f| < \min \{1/3 , 1/3\, T^n\} ,$$

then the equation (1) *has at least one* T–*periodic solution.*

Proof. For each $\varphi \in X$, define

$$F(\varphi)(t) = f(t, \varphi(t)) - \frac{1}{T} \int_0^T f(s, \varphi(s))\, ds . \tag{4}$$

Then $F(\varphi) \in X$ and $\int_0^T F(\varphi)(u)\, du = 0$. For each $\varphi \in X$ satisfying $\int_0^T \varphi(u)\, du = 0$, define

$$A(\varphi)(t) = \int_0^t \varphi(u)\, du - \frac{1}{T} \int_0^T \int_0^s \varphi(u)\, du\, ds . \tag{5}$$

Is is easy to see that $A(\varphi) \in X$, $\int_0^T A(\varphi)(u)\, du = 0$, $A(\varphi)'(t) = \varphi(t)$, and $\| A(\varphi) \| \leq T \| \varphi \|$. Also, for each $\varphi \in X$ satisfying $\int_0^T \varphi(u)\, du = 0$, define

$$B(\varphi)(t) = \int_0^t \varphi(u)\, du . \tag{6}$$

Then $B(\varphi) \in X$, $B(\varphi)'(t) = \varphi(t)$, and $\| B(\varphi) \| \leq (T/2) \| \varphi \|$. Write $A^m(\varphi) = A[A(\cdots A(\varphi)\cdots)]$, repeating m times, for any positive integer m. Hence, for $n \geq 2$ and for any $\varphi \in X$ satisfying $\int_0^T \varphi(u)\,du = 0$, we have $B[A^{n-1}(\varphi)] \in X$, $B[A^{n-1}(\varphi)]^{(n)}(t) = \varphi(t)$, and $\| B[A^{n-1}(\varphi)] \| \leq (T^n/2) \| \varphi \|$.

Let R denote the set of real numbers. For any $\lambda_i \in R$ and $(\varphi_i, r_i) \in X \times R$, $i = 1, 2$, let

$$\lambda_1(\varphi_1, r_1) + \lambda_2(\varphi_2, r_2) = (\lambda_1 \varphi_1 + \lambda_2 \varphi_2, \lambda_1 r_1 + \lambda_2 r_2).$$

Also, for each $(\varphi, r) \in X \times R$, define $|(\varphi, r)| = \|\varphi\| + |r|$. Then $X \times R$ becomes a Banach space.

Suppose that $xf(t, x) \geq 0$ for $|x| \geq M$ and for all t. Define a mapping $P : X \times R \to X \times R$ by $P(\varphi, r) = (\tilde{\varphi}, \tilde{r})$ with

$$\tilde{\varphi} = r + B[A^{n-1}(p - F(\varphi))], \tag{7}$$

$$\tilde{r} = r - \frac{1}{T}\int_0^T f(s, \tilde{\varphi}(s))\,ds. \tag{8}$$

Then P is a continuous mapping.

Since the quasinorm $|f| < \min\{1/3, 1/3\,T^n\}$, there is an $\varepsilon > 0$ such that $|f| + \varepsilon < \min\{1/3, 1/3\,T^n\}$. By the definition of quasiboundedness (3), there exists $\rho(\varepsilon) > 0$ such that

$$\frac{|f(t, x)|}{|x|} < |f| + \varepsilon \quad \text{whenever} \quad |x| \geq \rho(\varepsilon) \quad \text{and} \quad 0 \leq t \leq T.$$

Let

$$L = \max\{|f(t, x)| \mid 0 \leq t \leq T, |x| \leq \rho(\varepsilon)\},$$

$$N = \max\left\{\frac{M}{1 - 3(|f| + \varepsilon)}, \frac{M + (3/2)\,T^n \|p\|}{1 - 3(|f| + \varepsilon)\,T^n}, \frac{L}{|f| + \varepsilon}, \rho(\varepsilon)\right\},$$

and

$$C = \max\left\{(|f| + \varepsilon)\,N, \frac{1}{2}\,T^n \|p\| + (|f| + \varepsilon)\,T^n N\right\}.$$

Note that $M + 3C \leq N$ and

$$|f(t, x)| \leq (|f| + \varepsilon)\,N \quad \text{whenever} \quad |x| \leq N \quad \text{and} \quad 0 \leq t \leq T.$$

Now, let

$$D = \{(\varphi, r) \in X \times R \mid \|\varphi\| \leq N, |r| \leq M + 2C\}.$$

Then D is a closed, bounded, and convex set in $X \times R$. It is easy to show that $P(D) \subset D$ and $P(D)$ is relatively compact. Then by Schauder's fixed point Theorem ([9], or see [2, p. 131]) there exists $(\psi, b) \in D$ such that

$(\psi, b) = P(\psi, b) = (\tilde{\psi}, \tilde{b})$. It follows from (7) and (8) that $\frac{1}{T}\int_0^T f(s, \psi(s))\, ds = 0$ and hence by (4) we have $F(\psi)(t) = f(t, \psi(t))$. Differentiating the equation $\psi = b + B[A^{n-1}(p - F(\psi))]$ n times, we obtain

$$\psi^{(n)}(t) + f(t, \psi(t)) = p(t).$$

If $xf(t, x) \leq 0$ for $|x| \geq M$ and for all t, we redefine the $\tilde{r}$ in (8) as

$$\tilde{r} = r + \frac{1}{T}\int_0^T f(s, \tilde{\varphi}(s))\, ds.$$

Then the same argument as before leads to the desired result. This completes the proof.

COROLLARY 2.2. *Let f be continuous and $f(t + T, x) = f(t, x)$ for all (t, x) and for some $T > 0$, p continuous and $p(t + T) = p(t)$ for all t, and $\int_0^T p(u)\, du = 0$. Assume that there is a positive number M such that for $|x| \geq M$ and for all t either $xf(t, x) \geq 0$ or $xf(t, x) \leq 0$. If $(f(t, x)/x) \to 0$ as $|x| \to \infty$ uniformly in t, then the equation (1) has at least one T-periodic solution.*

Proof. The condition $(f(t, x)/x) \to 0$ as $|x| \to \infty$ uniformly in t implies that $|f| = 0$.

Remark. The above corollary extends a result in [1] which in turn generalizes a result of Lazer [4] for the case $f(t, x) = f(x)$ and $n = 2$.

3. EVEN AND ODD SOLUTIONS

Let

$$Y = \{\varphi \in X \mid \varphi(-t) = \varphi(t) \text{ for all } t\}$$

and

$$Z = \{\varphi \in X \mid \varphi(-t) = -\varphi(t) \text{ for all } t\}.$$

Then both Y and Z are also Banach spaces under the supremum norm.

THEOREM 3.1. *Let n be an even integer ≥ 2. Assume that $f(-t, x) = f(t, x)$ for any (t, x) and p is even.* If *f is quasibounded with a quasinorm*

$$|f| < \min\{1/3, (1/3)(\sqrt{2}/T)^n\},$$

and if all other conditions of Theorem 2.1 are satsified, then the equation (1) has at least one even T-periodic solution.

Proof. For each $\varphi \in Y$, define $F(\varphi)$ as in (4). Then $F(\varphi) \in Y$ and $\int_0^T F(\varphi)(u)\,du = 0$. Also, for each $\varphi \in Y$ satisfying $\int_0^T \varphi(u)\,du = 0$, define $E(\varphi) = A[B(\varphi)]$, where A and B are defined as in (5) and (6). Then $E(\varphi) \in Y$, $\int_0^T E(\varphi)(u)\,du = 0$, $E(\varphi)''(t) = \varphi(t)$, and $\| E(\varphi) \| \leq (T^2/2) \| \varphi \|$.

Suppose that $xf(t, x) \geq 0$ for $|x| \geq M$ and for all t. Define $Y \times R$ similarly as in the proof of Theorem 2.1 and a mapping $P_1 : Y \times R \to Y \times R$ by $P_1(\varphi, r) = (\tilde{\varphi}, \tilde{r})$ with

$$\tilde{\varphi} = r + B^2 [E^{(n-2)/2}(p - F(\varphi))],$$

$$\tilde{r} = r - \frac{1}{T} \int_0^T f(s, \tilde{\varphi}(s))\,ds.$$

Clearly this is well-defined. Then one completes the proof by an argument similar to that in the proof of Theorem 2.1.

THEOREM 3.2. *Let n be an even integer ≥ 2. Let f be continuous and $f(t + T, x) = f(t, x)$ for all (t, x) and for some $T > 0$, and p continuous and $p(t + T) = p(t)$ for all t. Let p be odd and f satisfy either (i) $f(-t, x) = f(t, x)$ and $f(t, -x) = -f(t, x)$, or (ii) $f(-t, x) = -f(t, x)$ and $f(t, -x) = f(t, x)$, for any (t, x). If f is quasibounded with a quasinorm $|f| < (\sqrt{2}/T)^n$, then the equation (1) has at least one odd T-periodic solution.*

Proof. For each $\varphi \in Z$, let $\hat{f}(\varphi)(t) = f(t, \varphi(t))$. Then for both cases (i) and (ii) we have $\hat{f}(\varphi) \in Z$. Also, for each $\varphi \in Z$, define $G(\varphi) = B[A(\varphi)]$. Then $G(\varphi) \in Z$, $G(\varphi)''(t) = \varphi(t)$, and $\| G(\varphi) \| \leq (T^2/2) \| \varphi \|$. Now, define a mapping $P_2 : Z \to Z$ by $P_2(\varphi) = G^{n/2}(p - \hat{f}(\varphi))$.

By the assumption on the quasinorm of f, there exists $\varepsilon > 0$ such that

$$\frac{|f(t, x)|}{|x|} < |f| + \varepsilon < \left(\frac{\sqrt{2}}{T}\right)^n \quad \text{whenever } |x| \geq \rho(\varepsilon) \text{ and } 0 \leq t \leq T$$

for some $\rho(\varepsilon) > 0$. Let

$$L = \max \{ |f(t, x)| \mid 0 \leq t \leq T, |x| \leq \rho(\varepsilon) \}$$

and

$$N = \max \left\{ \frac{T^n \| p \|}{2^{n/2} - (|f| + \varepsilon) T^n}, \frac{L}{|f| + \varepsilon}, \rho(\varepsilon) \right\}.$$

Note that $|f(t, x)| \leq (|f| + \varepsilon) N$ whenever $|x| \leq N$ and $0 \leq t \leq T$. Let

$$D = \{ \varphi \in Z \mid \| \varphi \| \leq N \}.$$

It is easy to show that $P_2(D) \subset D$ and $P_2(D)$ is relatively compact. The result then follows from Schauder's fixed point Theorem.

THEOREM 3.3. *Let n be an odd integer ≥ 1. Let f be continuous and $f(t+T, x) = f(t, x)$ for all (t, x) and for some $T > 0$, and p continuous and $p(t+T) = p(t)$ for all t. Assume that $f(-t, x) = -f(t, x)$ for any (t, x) and p is odd. If f is quasibounded with a quasinorm $|f| < \sqrt{2}(\sqrt{2}/T)^n$, then the equation* (1) *has at least one even T–periodic solution.*

Proof. For each $\varphi \in Y$, let $\hat{f}(\varphi)(t) = f(t, \varphi(t))$. Then here we have $\hat{f}(\varphi) \in Z$. Define a mapping $P_3 : Y \to Y$ by

$$P_3(\varphi) = B[p - \hat{f}(\varphi)], \quad \text{if} \quad n = 1,$$

and

$$P_3(\varphi) = B^2[E^{(n-3)/2}(A(p - \hat{f}(\varphi)))], \quad \text{if} \quad n \geq 3,$$

where E is defined as in the proof of Theorem 3.1. The rest of the proof is similar to that of Theorem 3.2 and is therefore omitted.

Remark. The technique used in this section does not produce a similar result on the existence of odd T–periodic solutions when n is an odd integer.

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