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Topologia. — *Lattice Theory and Jacobson Rings*. Nota di CHARLES SUFFEL, EDWARD BECKENSTEIN e LAWRENCE NARICI, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Viene studiato il completamento di Jacobson di uno spazio topologico T_0, e ne vengono fatte applicazioni allo studio degli anelli comutativi con identità.

SECTION 0. INTRODUCTION

Let T be a T_0 space and L a lattice of closed subsets of T which form a base for the closed sets in T. If T were a T_1 space, the collection W (T , L) of L–ultrafilters could be topologized so as to form a T_1–compactification of T referred to as a Wallman-type compactification of T. One reason for interest in such compactifications is the question of whether an arbitrary Hausdorff compactification of a Tychonoff space T can be realized as a Wallman-type compactification. This question remains open.

We study here a larger compact space, J (T , L), than W (T , L) (Def. 5). W (T , L) is very dense in J (T , L) (for any closed set F in J (T , L), $\mathrm{cl}_J\, F \cap W(T, L) = F$), and for every irreducible closed set F in W (T , L), J (T , L) contains a generic point for F (a point x such that $\mathrm{cl}_J\, x = \mathrm{cl}_J\, F$). J(T, L) is called a *Jacobson completion* of T. It is shown to exist in a number of forms and to be unique when T is compact. Another approach to Jacobson completions using other techniques can be found in [4].

The material on Jacobson completions is applied to the study of commutative rings A with identity. Let $\mathcal{M}(A)$ be the maximal ideals of A and $\mathcal{J}(A)$ the Jacobson prime ideals (those prime ideals p such that ($p = \bigcap_{M \supset p, M \in \mathcal{M}(A)} M$). It is shown that $\mathcal{J}(A) = J(\mathcal{M}(A), L)$ where L is either of two lattices of hull-kernel closed subsets of $\mathcal{M}(A)$. As a consequence, $\mathcal{J}(A)$ is the Jacobson completion of $\mathcal{M}(A)$ and the generic points of irreducible closed subsets of $\mathcal{M}(A)$ lie in $\mathcal{J}(A)$. This generalizes some results of [2] and [3].

SECTION 1. VERY DENSE SPACES

In this section we develop some topological relationships between a space Y and a very dense subspace X. X is *very dense* if Y if and only if $\mathrm{cl}_Y (F \cap X) = F$ for any closed set $F \subset Y$. We show that if X is a compact T_1 space, then X can be extended to a T_0 space Y in which X is very dense and every irreducible closed subset of X has a generic point in the sense of Prop. 7. If in X the compact-open sets form a base for the topology closed with respect to the formation of finite intersections, then this is shown to be true in Y as

(*) Nella seduta del 14 novembre 1974.

well. Hochster in [5] referred to a space such as Y as a *spectral space*, and we will refer to X as a *prespectral space*.

Only brief sketches of proofs will be presented in this section.

PROPOSITION 1. X *is very dense in* Y *if and only if for each* $y \in Y$, $y \in \mathrm{cl}_Y (\mathrm{cl}_Y \{y\} \cap X)$.

PROPOSITION 2. (*a*) *If* X *is very dense in* Y *and* $\{B_\alpha\}$ *is a base for the closed subsets of* X, *then* $\{\mathrm{cl}_Y B_\alpha\}$ *is a base for the closed subsets of* Y.

(b) *If* X *is very dense in* Y *and* $\{F_\beta\}$ *is a family of closed subsets of* X, *then* $\mathrm{cl}_Y \cap F_\beta = \cap\, \mathrm{cl}_Y F_\beta$.

PROPOSITION 3. (a) *If* X *is very dense in* Y, *a closed set* $F \subset X$ *is irreducible if and only if* $\mathrm{cl}_Y F$ *is irredubible in* Y.

(b) $\mathrm{cl}_Y \{y\}$ *is irreducible for each* $y \in Y$.

(c) *Letting* Irr X *denote the irreducible closed subsets of* X,

$$I : Y \longrightarrow \mathrm{Irr}\, X$$

$$y \longrightarrow \mathrm{cl}_Y \{y\} \cap X$$

is a 1 — 1 *mapping*.

PROPOSITION 4. (a) *If* X *is very dense in* Y *and* U *is open in* X, *then there exists a unique open set* $\hat{U} \subset Y$ *such that* $\hat{U} \cap X = U$.

(b) *The set* U *of* (a) *is compact if and only if* $\hat{U}$ *is compact*.

Proof. (a) $C\hat{U} \cap X = CU$ and X is very dense in Y. Thus $C\hat{U}$ is unique.

(b) If F is any closed set in X and U any open set in X, show that U meets F if and only if $\hat{U}$ meets F where $\hat{U}$ is the set of (a). Then show that a family of closed subsets of $\hat{U}$ with the finite intersection property has nonempty intersection. Use Prop. 2 (b).

DEFINITION 1. A prespectral *space* X *is a compact space such that the compact-open sets form a base for the topology which is closed with respect to the formation of finite intersections*.

PROPOSITION 5. *If* X *is very dense in* Y, *then* X *is prespectral if and only if* Y *is prespectral*.

DEFINITION 2. *A point* p *is said to be adjoined to* X *if* X *is very dense in* $X \cup \{p\}$.

By Prop. 3 (b) it is clear that adjoining a point to X amounts to adding a generic point for an irreducible closed subset of X (necessarily containing more than one point). We develop a proceedure for adjoining them all.

PROPOSITION 6. *If* X *is a* T_1 *prespectral space, then there exists a point* p *such that* p *can be adjoined to* X *if and only if there exists a filterbase of compact-open sets* $\mathscr{B}$ *such that* $\cap \mathscr{B} = \varnothing$.

Proof. Start by extending $\mathscr{B}$ to an ultrafilter among the compact-open subsets of X. Refer to this ultrafilter again as $\mathscr{B}$. We define a topology on the set $X \cup \{p\}$ as follows

(1) If $N \supset B$ for some $B \in \mathscr{B}$, then $\{p\} \cup N$ is a neighborhood of p.

(2) If $x \in X$ and N is a neighborhood of x in X, then if $N \supset B$ for some $B \in \mathscr{B}$, $\{p\} \cup N$ is a neighborhood of x.

(3) If $x \in X$ and N is a neighborhood of x in X containing no set $B \in \mathscr{B}$, then N is a neighborhood of x again in $\{p\} \cup X$.

PROPOSITION 7. *If* X *is a* T_1 *space and* F *an irreducible closed set with more than one point, then a generic point* P *for* F *can be adjoined to* X (*i.e.* $(\mathrm{cl}_{X \cup \{p\}} \{P\}) \cap X = F$).

Proof. Let $\mathscr{B}$ be the collection of open subsets of X which meet F. As F is irreducible, $\mathscr{B}$ is a filter. As X is a T_1 space, $\cap \mathscr{B} = \varnothing$. We define neighborhoods of points in the space $X \cup \{p\}$ exactly as in the previous result.

PROPOSITION 8. *If* X *is a* T_1 *space, then* X *can be extended to a space* Y *in which* X *is very dense and every irreducible closed subset of* X *has a generic point in the sense of Prop. 7.*

Proof. Let $\mathscr{S}$ be the irreducible closed subsets with no generic points in X. Let $T = X \cup Z$ with the cardinality of Z strictly greater than the cardinality of $\mathscr{S}$. We consider the family $\mathscr{A}$ of topological spaces such that for each $S \in \mathscr{A}$, $X \subset S \subset T$ and X is very dense in S. We order $\mathscr{A}$ under the relationship $S_1 \leqq S_2$ if and only if S_1 is a subspace of S_2. It follows then that S_1 is very dense in S_2. It can be shown that $\mathscr{A}$ is inductively ordered and contains a maximal element S_M. We know (Prop. 3 (c)) that S_M consists of generic points of irreducible closed subsets of X. It can be shown that S_M is the space Y of the theorem as follows: Since $\overline{\overline{Z}} > \overline{\overline{\mathscr{S}}}$, S_M cannot have exhausted Z. If there is some $F \subset X$ such that $F \in \mathscr{S}$ and F has no generic point in S_M, we adjoin a point $z \in Z$ to S_M as follows.

Let $\mathscr{B}$ be the filter of open subsets of X associated with F as in Prop. 7. For each $O_\alpha \in \mathscr{B}$ let $\hat{O}_\alpha$ be the unique subset of S_M such that $\hat{O}_\alpha \cap X = O_\alpha$. We adjoin z to S_M by setting $Y = S_M \cup \{z\}$ and defining neighborhoods of points in Y by

(1) $\{z\} \cup N$ is a neighborhood of z in Y if for some $\hat{O}_\alpha$, $\hat{O}_\alpha \subset N \subset S_M$.

(2) If $s \in S_M$ and N is a neighborhood of s such that for some $\hat{O}_\alpha$, $s \in \hat{O}_\alpha \subset N$, then $\{z\} \cup N$ is a neighborhood of s.

(3) If N is a neighborhood of s in S_M and there exists no $\hat{O}_\alpha$ such that $s \in \hat{O}_\alpha \subset N$, then N remains a neighborhood of s in Y.

It can be shown that S_M and X are both very dense in Y which violates the maximality of S_M in T. There are numerous elementary steps in the verification of the statements of the previous sketch. These are left to the reader

DEFINITION 3. (a) *If* X *is a* T_1 *space, a* Jacobson completion *of* X *is a space* Y *in which* X *is very dense and every irreducible closed subset of* X *has a generic point.*

(b) *If* X *is* T_1 *and prespectral, a Jacobson completion of* X *is called a* spectral completion.

PROPOSITION 9. *A Jacobson completion exists for every* T_1 *space* X.

Proof. See Prop. 8

DEFINITION 4. *A* T_1 *space* X *is* spectrally complete *if and only if it is prespectral and admits no proper Jacobson completion.*

PROPOSITION 10. *A prespectral* T_1 *space* X *is spectrally complete if and only if any of the following are true.*

(a) *Each filterbase of compact-open subsets of* X *has nonempty intersection.*

(b) X *is not very dense in any proper extension* Y.

(c) X *contains the generic points of all irreducible closed sets.*

In [5] Hochster has shown that a prespectral space which is spectrally complete is topologically equivalent to the prime ideals of a ring. He called such a space a spectral space. We have shown that every T_1 prespectral space X can be enlarged to a maximal spectral space Y in which it is very dense.

By the results of Section 3 (Prop. 17) it will be seen that Y constitutes the prime ideals of a Jacobson ring for which X constitutes the maximal ideals. Hence every prespectral T_1 space X is the maximal ideals of a Jacobson ring. In Section 3 (Prop. 13), it will emerge that the converse of this is also true.

SECTION 2. LATTICES AND THE JACOBSON COMPLETION

In this section we essentially, reproduce the material of Section 1 utilizing lattice theory. We show that any Jacobson completion of a compact T_1 space X is a Wallman type compactification of X and is unique (Prop. 12). Compactness of W was not assumed in Prop. 8. As we are most interested in the applications of these results to ring theory in which the compact T_1 space $\mathcal{M}(A)$ of maximal ideals of the ring A plays the role of X, we donot regard this as a serious drawback.

DEFINITION 5. *Let* L *be a distributive lattice with* 0 *and* 1. *A prime filter* $\mathcal{P}$ *in* L *is a filter such that if* $a, b \in L$ *and* $a + b = \mathcal{P}$, *then* $a \in \mathcal{P}$ *or* $b \in \mathcal{P}$. *A* Jacobson filter $\mathcal{J}$ *is a prime filter such that* $\mathcal{J}$ *is the intersection of all the ultrafilters containing it. We adopt the following notations.*

W(L)—*the set of all ultrafilters*;
J (L)—*the set of Jacobson filters*;
P (L)—*the set of prime filters.*

If $a \in L$, then we set $\beta_a = \{ \mathcal{P} \in P(L) / a \in \mathcal{P} \}$. On P(L) it is readily shown that $\beta_{a+b} = \beta_a \cup \beta_b$ and $\beta_{ab} = \beta_a \cap \beta_b$. Hence the sets $\{ \beta_a / a \in L \}$ are a base for the closed sets of a topology on P(L) which we refer to as the Wallman topology.

If we restrict our attention to $W(L)$ we find that $C\beta_\alpha \cap W(L) = \{\mathscr{L} \in W(L) / \text{there exists } b \in \mathscr{L} \text{ with } ab = 0\}$. This is no longer true once we leave W(L) and in fact if $\mathscr{P} \in P(L) - W(L)$, then as $\mathscr{P}$ can be extended to an ultrafilter $\mathscr{L}$ and letting $a \in \mathscr{L} - \mathscr{P}$, then $\mathscr{P} \in C\beta_\alpha$ but $ab \neq 0$ for all $b \in \mathscr{P}$.

From this point on in the section we assume that X is a T_1 space and $\mathscr{C}$ the lattice of all closed subsets of X. The spaces $W(\mathscr{C})$, $J(\mathscr{C})$, and $P(\mathscr{C})$ will be denoted by $W(X, \mathscr{C})$, $J(X, \mathscr{C})$, and $P(X, \mathscr{C})$ respectively.

PROPOSITION 11. (a) *A filter* $\mathscr{P}_F = \{K \in \mathscr{C} / F \subset K\}$ *where* F *is closed, is a prime filter if and only if* F *is irreducible*

(b) *When* X *is compact, a filter* $\mathscr{L} \in W(X, \mathscr{C})$ *if and only if* $\mathscr{L} = \mathscr{L}_x = \{K \in \mathscr{C} / x \in K\}$ *for some* $x \in X$.

(c) *When* X *is compact, a filter* $\mathscr{J} \in J(X, \mathscr{C})$ *if and only if* $\mathscr{J} = \mathscr{P}_F$ *for some irreducible closed set* $F \subset X$.

Proof. (a) $\mathscr{P}_F$ is prime if and only if when $F \subset F_1 \cup F_2$, then $F \subset F_1$ or $F \subset F_2$, that is, if and only if F is irreducible.

(b) Suppose X is compact and $\mathscr{L}$ is an ultrafilter. Then $\bigcap_{\mathscr{L}} K \neq \emptyset$. Thus for some $x \in X$, $x \in \bigcap_{\mathscr{L}} K$ and it readily follows that $\mathscr{L} = \mathscr{L}_x$.

(c) Suppose X is compact. Let $\mathscr{J} \in J(X, \mathscr{C})$. Then $\mathscr{J} = \bigcap_{\mathscr{J} \subset \mathscr{L}_x} \mathscr{L}_x$ and let $F = \{x \in X / \mathscr{J} \subset \mathscr{L}_x\}$. Then $cl_X F \subset \mathscr{L}_x$ for all $\mathscr{L}_x$ such that $\mathscr{J} \subset \mathscr{L}_x$ and $cl_X F \in \mathscr{J}$. If $K \in \mathscr{J}$, then $K \in \mathscr{L}_x$ for all x such that $\mathscr{J} \subset \mathscr{L}_x$. Hence $F \subset cl_X F \subset K$ and it follows that $\mathscr{J} \subset \{H \in \mathscr{C} / cl_X F \subset H\}$ However, as $cl_X F \in \mathscr{J}$, it follows that $\{H \in \mathscr{C} / cl_X F \subset H\} \subset \mathscr{J}$ and therefore that $\mathscr{J} = \mathscr{P}_{cl_X F}$. Clearly then $F = cl_X F$ and by (a), Fis irreducible.

Conversely, if $\mathscr{J} = \mathscr{P}_F$ where F is an irreducible closed set, then $\mathscr{J} = \mathscr{P}_F = \bigcap_{\mathscr{J} \subset \mathscr{L}_x} \mathscr{L}_x$ and clearly $\mathscr{J} \in J(X, \mathscr{C})$.

PROPOSITION 12. *Let* X *be a compact* T_1 *space and* X *very dense in* Y. *Then with* $F_y = cl_Y \{y\} \cap X$,

$$\sigma : Y \longrightarrow J(X, \mathscr{C})$$
$$y \longrightarrow \mathscr{P}_{F_y}$$

is a homeomorphism such that σ *restricted to* X *establishes a homeomorphism between* X *and* $W(X, \mathscr{C})$. σ *is onto* $J(X, \mathscr{C})$ *if and only if every irreducible closed set* $F \subset X$ *has a generic point in* Y.

Proof. As $F_y = (cl_Y \{y\}) \cap X$ is irreducible, $\sigma(y) = \mathscr{P}_{F_y}$ $J(X, \mathscr{C})$. As Y is a T_0 space, $cl_Y \{y_1\} \neq cl_Y \{y_2\}$ if $y_1 \neq y_2$ and as X is very dense in Y, is follows that $F_{y_1} \neq F_{y_2}$ and σ is a 1 — 1 mapping.

By Prop. 11 (b), $\sigma(y) \in W(X, \mathscr{C})$ if and only if $\sigma(y) = \mathscr{P}_{F_y} = \mathscr{L}_x = \sigma(x)$. Thus $\sigma(X) = W(X, \mathscr{C})$.

Suppose now that σ is onto. Then for each irreducible closed set $F \subset X$, $\mathscr{P}_F = \{K \in \mathscr{C} / F \subset K\} \in J(X, \mathscr{C})$ and there exists y such that $\mathscr{P}_F = \mathscr{P}_{F_y}$. Hence $F = F_y$.

Conversely it is clear that if for each irreducible closed set $F \subset X$, $F = F_y$ for some $y \in Y$, then σ is an onto map.

To show that σ is a homeomorphism we simply note that if F is closed in Y, then $\sigma(F) = \{ \mathscr{P}_{F_y} / y \in F \} = \{ \mathscr{P}_{F_y} / F \cap X \in \mathscr{P}_{F_y} \} = \beta_{F \cap X} \cap J(X, \mathscr{C})$.

COROLLARY. *If* X *is a compact* T_1 *space, the Jacobson compactification of* X, *exists, is unique, and is equivalent to* $J(X, \mathscr{C})$.

COROLLARY. *If* X *is a prespectral* T_1 *space,* $J(X, \mathscr{C})$ *is the spectral completion of* X.

SECTION 3. APPLICATIONS TO RING THEORY.

In this section we apply the material of the previous two sections to relationships between the prime and maximal ideals of a commutative ring with identity. Denoting the maximal ideals of A as $\mathscr{M}(A)$ and the hull of $\{a_1, \cdots, a_n\} \subset A$ as $H_{\mathscr{M}(A)}(a_1, \cdots, a_n) = \{M \in \mathscr{M}(A) / a_i \in M\}$, $L_{\mathscr{M}(A)}$ as, the lattice of all such hulls, $\mathscr{J}(A)$ as the set of all Jacobson prime ideals, we show that if $\mathscr{C}$ is the lattice of all bull-kernel closed subsets of $\mathscr{M}(A)$, then $\mathscr{J}(A) = J(\mathscr{M}(A), L_{\mathscr{M}(A)}) = J(\mathscr{M}(A), \mathscr{C})$.

In [3] Grothedieck showed that the points of Spec A (the set of all prime ideals of A) can be put in 1 — 1 correspondence with the irreducible closed subsets of Spec A under the mapping $p \to \mathrm{cl}_{\mathrm{Spec}\,A}\{p\}$. In [2] it was shown that if A is a Jacobson ring ($\mathscr{J}(A) = \mathrm{Spec}\,A$), this correspondence can be established between the points of SpecA and the irreducible closed sets in $\mathscr{M}(A)$ under the mapping $p \to (\mathrm{cl}_{\mathrm{Spec}\,A}\{p\}) \cap \mathscr{M}(A)$. Critical in proving the result is the fact $\mathscr{M}(A)$ is very dense in Spec A when A is a Jacobson ring. We are interested in locating the generic points of the irreducible closed subsets of $\mathscr{M}(A)$ when A is not a Jacobson ring. Here, having shown (Prop. 15) that $\mathscr{J}(A)$ is the Jacobson completion of $\mathscr{M}(A)$, we find (Prop. 16) that these generic points are located in $\mathscr{J}(A)$.

In addition, we prove (Prop. 17) that a topological space X is identifiable as the maximal ideals of a Jacobson ring A if and only if X is a prespectral T_1 space. In such a case it is shown that the Jacobson completion of X is identifiable as SpecA.

DEFINITION 6. *Let* $S \subset \mathrm{Spec}\,A$. *Then*

$$H_S(a_1, \cdots, a_n) = \{ p \in S / a_i \in p \}$$

$$L_S = \{ H_S(a_1, \cdots, a_n) / a_i \in A \}.$$

The sets $H_S(a_1, \cdots, a_n)$ are a base for a topology on S referred to as the hull-kernel topology. Any closed subset of S in the hull-kernel topology is of the form $\mathrm{cl}_S F = H_S(kF) = \{ p \in S / kF \subset p \}$ where $F \subset S$ and $kF = \bigcap_{p \in F} p$. The space $\mathscr{M}(A)$ is a compact T_1 space and SpecA is a compact T_0 space.

PROPOSITION 13. (a) *If* $p \in S$, *then* $\mathrm{cl}_S\{p\} = \{p' \in S / p \subset p'\}$.

(b) *If* $F \subset S$, $\mathrm{cl}_S F = \{p \in S / kF \subset p\}$.

(c) *If* $\mathcal{M}(A) \subset S \subset \mathrm{Spec}\, A$, *then* S *is compact.*

(d) *If* $\mathcal{M}(A) \subset S \subset \mathrm{Spec}\, A$, *then* $\mathcal{M}(A)$ *is very dense in* S *if and only if* $S \subset \mathcal{J}(A)$.

Proof. (d) If $p \in \mathcal{J}(A)$, then $\mathrm{cl}_S\{p\} \cap \mathcal{M}(A) = \{M \in \mathcal{M}(A) / M \supset p\}$. As $p = \bigcap_{p \subset M} M$, if $p \in \mathrm{CH}_S(a_1, \cdots, a_n)$, then $a_i \notin p$ for some i. Hence there exists $M \in \mathcal{M}(A)$ such that $p \subset M$ and $a_i \notin M$ for some i. Thus $\mathrm{CH}_S(a_1, \cdots, a_n) \cap \cap\, \mathrm{cl}_S\{p\} \cap \mathcal{M}(A) \neq \varnothing$ and $p \in \mathrm{cl}_S(\mathrm{cl}_S\{p\} \cap \mathcal{M}(A))$. Hence by Prop. 1, $\mathcal{M}(A)$ is very dense in S under the assumption $S \subset \mathcal{J}(A)$.

Conversely if $p \in S$ and $p \notin \mathcal{J}(A)$, then $p \neq \bigcap_{p \subset M} M$ and there exists $a \in \bigcap_{p \subset M} M$ with $a \notin p$. Hence $p \in \mathrm{CH}_S(a)$ but for any M such that $M \supset p$, $M \in \mathrm{H}_S(a)$. Thus $\mathrm{CH}_S(a) \cap \mathrm{cl}_S\{p\} \cap \mathcal{M}(A) = \varnothing$.

DEFINITION 7. *Let* S *be such that* $\mathcal{M}(A) \subset S \subset \mathrm{Spec}\, A$. *Then* S *satisfies condition* H_S *if* $\mathrm{cl}_S \mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) = \mathrm{H}_S(a_1, \cdots, a_n)$ *for any* $\{a_1, \cdots, a_n\} \subset A$.

PROPOSITION 14. (*a*) *If* $\mathcal{M}(A) \subset S \subset \mathcal{J}(A)$, *then* S *satisfies condition* H_S.

(b) *If* $\mathcal{M}(A) \subset S \subset \mathcal{J}(A)$ *and* F *is a closed subset of* $\mathcal{M}(A)$, *then* $\mathrm{cl}_S F = \{p \in S / kF \subset p\}$ *and every closed set in* $\mathcal{J}(A)$ *is of this form.*

Proof. (a) *If* $\mathcal{M}(A) \subset S \subset \mathcal{J}(A)$, then as $\mathcal{M}(A)$ is very dense in S and $\mathrm{H}_S(a_1, \cdots, a_n) \cap \mathcal{M}(A) = \mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n)$, the proof is done.

(b) Let F be a closed subset of $\mathcal{M}(A)$. Then $F = \bigcap_{a_\alpha \in kF} \mathrm{H}_{\mathcal{M}(A)}(a_\alpha)$. As $\mathcal{M}(A) \subset S \subset \mathcal{J}(A)$, then $\mathcal{M}(A)$ is very dense in S and S satisfies condition H_S. Then $\mathrm{cl}_S F = \bigcap_{a_\alpha \in kF} \mathrm{cl}_S \mathrm{H}_{\mathcal{M}(A)}(a_\alpha) = \bigcap_{a_\alpha \in kF} \mathrm{H}_S(a_\alpha)$. Hence $p \in \mathrm{cl}_S F$ if and only if for all $a_\alpha \in kF$, $p \in \mathrm{H}_S(a_\alpha)$. Equivalently, $p \in \mathrm{cl}_S F$ if and only if $kF \subset p$. As $\mathcal{M}(A)$ is very dense in S, the remainder of the statement of (b) follows immediately.

We establish notation here which is in force for the remainder of the paper. Let $p \in \mathrm{Spec}\, A$. Then $\sigma(p) = \{\mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) / a_i \in p\}$. Clearly by Prop. 14, $\sigma(p) = \{\mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) / p \in \mathrm{H}_{\mathcal{J}(A)}(a_1, \cdots, a_n) = \mathrm{cl}_{\mathcal{J}(A)} \mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n)\}$ for all $p \in \mathcal{J}(A)$. If $p \in \mathrm{Spec}\, A$, then $\hat{\sigma}(p) = \{F \in \mathscr{C} / kF \subset p\}$. If $p \in \mathcal{J}(A)$, then by Prop. 14 (b) $\hat{\sigma}(p) = \{F \in \mathscr{C} / p \in \mathrm{cl}_{\mathcal{J}(A)} F\}$.

PROPOSITION 15. (a) *The Mapping* σ *establishes a homeomorphism between* $\mathcal{J}(A)$ *and* $\mathrm{J}(\mathcal{M}(A), \mathrm{L}_{\mathcal{M}(A)})$ *with* $\sigma(\mathcal{M}(A)) = \mathrm{W}(\mathcal{M}(A), \mathrm{L}_{\mathcal{M}(A)})$.

(b) *The mapping* $\hat{\sigma}$ *establishes a homeomorphism between.* $\mathcal{J}(A)$ *and* $\mathrm{J}(\mathcal{M}(A), \mathscr{C})$ *with* $\hat{\sigma}(\mathcal{M}(A)) = \mathrm{W}(\mathcal{M}(A), \mathscr{C})$.

Proof. (a) We first show that if $p \in \mathcal{J}(A)$, then $\sigma(p) \in \mathrm{J}(\mathcal{M}(A), \mathrm{L}_{\mathcal{M}(A)})$. If $\mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) \subset \mathrm{H}_{\mathcal{M}(A)}(b_1, \cdots, b_m)$ with $\mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) \in \sigma(p)$, then as $p \in \mathrm{cl}_{\mathcal{J}(A)} \mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) = \mathrm{H}_{\mathcal{M}(A)}(a_1, \cdots, a_n) \subset \mathrm{cl}_{\mathcal{J}(A)} \mathrm{H}_{\mathcal{M}(A)}(b_1, \cdots, b_m) = \mathrm{H}_{\mathcal{J}(A)}(b_1, \cdots, b_m)$ it follows that $\mathrm{H}_{\mathcal{J}(A)}(b_1, \cdots, b_m) \in \sigma(p)$ and clearly $\sigma(p)$ is a filter. As p is a prime ideal, it follows readily that $\sigma(p)$ is a prime filter.

To show that $\sigma(p)$ is a Jacobson filter, we first characterize the ultrafilters. We show that $\mathscr{X}$ is an ultrafilter if and only if $\mathscr{X} = \sigma(M)$ where M is a maximal ideal. If $H_{\mathscr{M}(A)}(a) \notin \sigma(M)$, then there exists $m \in M$ such that $ab + m = e$ for some $b \in A$. Then $H_{\mathscr{M}(A)}(a) \cap H_{\mathscr{M}(A)}(m) = \varnothing$ and it follows that $\sigma(M)$ is an ultrafilter. Conversely if $\mathscr{X}$ is an ultrafilter and $p = \{a \in A / H_{\mathscr{M}(A)}(a) \in \mathscr{X}\}$, it follows readily that p is a maximal ideal and that $\sigma(p) = \mathscr{X}$.

To complete the proof that when $p \in \mathscr{J}(A)$, $\sigma(p) \in J(\mathscr{M}(A), L_{\mathscr{M}(A)})$, we show that $\sigma(p) = \bigcap_{\sigma(p) \subset \sigma(M)} \sigma(M) = \bigcap_{p \subset M} \sigma(M)$. This follows readily from the fact that p is a Jacobson prime ideal for if $H_{\mathscr{M}(A)}(a_1, \cdots, a_n) \in \sigma(M)$ for all M such that $p \subset M$, then $\{a_1, \cdots, a_n\} \subset M$ for all M such that $p \subset M$. Thus $\{a_1, \cdots, a_n\} \subset p$ and therefore $H_{\mathscr{M}(A)}(a_1, \cdots, a_n) \in \sigma(p)$. Thus $\sigma(p) = \bigcap_{\mathscr{J} \subset \sigma(M)} \sigma(M)$.

To show that σ is onto, let $\mathscr{J} \in J(\mathscr{M}(A), L_{\mathscr{M}(A)})$. Then $\mathscr{J} = \bigcap_{\mathscr{J} \subset \sigma(M)} \sigma(M)$. Let $p = \{a \in A / H_{\mathscr{M}(A)}(a) \in \mathscr{J}\}$. Clearly p is a prime ideal and $\sigma(p) = \mathscr{J}$. It is clear that $\sigma(p) \subset \sigma(M)$ (M a maximal ideal) if and only if $p \subset M$. Thus to show that p is a Jacobson ideal we must show that $p = \bigcap_{p \subset M} M$. But if $a \in M$ for all M such that $p \subset M$, then $H_{\mathscr{M}(A)}(a) \in \sigma(M)$ for all $\sigma(M)$ such that $\mathscr{J} \subset \sigma(M)$. Therefore $H_{\mathscr{M}(A)}(a) \in \mathscr{J}$ and $a \in p$.

To show that σ is a 1 — 1 mapping we assume $p_1 \neq p_2$ and $a \in p_1$ with $a \notin p_2$. Clearly $H_{\mathscr{M}(A)}(a) \in \sigma(p_1)$ but if $H_{\mathscr{M}(A)}(a) \in \sigma(p_2)$, then $H_{\mathscr{M}(A)}(a) = H_{\mathscr{M}(A)}(a_1, \cdots, a_n)$ where $a_i \in p_2$. Thus $cl_{\mathscr{J}(A)} H_{\mathscr{M}(A)}(a) = H_{\mathscr{J}(A)}(a) = cl_{\mathscr{J}(A)} H_{\mathscr{M}(A)}(a_1, \cdots, a_n) = H_{\mathscr{J}(A)}(a_1, \cdots, a_n)$ and as $p_2 \in H_{\mathscr{J}(A)}(a_1, \cdots, a_n)$, $p_2 \in H_{\mathscr{J}(A)}(a)$ and $a \in p_2$ but this is a contradiction.

As $\sigma(H_{\mathscr{J}(A)}(a_1, \cdots, a_n)) = \{\mathscr{J} \in J(\mathscr{M}(A), L_{\mathscr{M}(A)}) / H_{\mathscr{M}(A)}(a_1, \cdots, a_n) \in \mathscr{J}\} = \beta_{H_{\mathscr{M}(A)}}(a_1, \cdots, a_n) \cap \mathscr{J}(A)$, and $\sigma^{-1}(\mathscr{J}(A) \cap H_{\mathscr{M}(A)}(a_1, \cdots, a_n)) = H_{\mathscr{J}(A)}(a_1, \cdots, a_n)$, then σ is clearly a homeomorphism.

(b) Since $\mathscr{M}(A)$ is very dense in $\mathscr{J}(A)$, if $p \in cl_{\mathscr{J}(A)} F$ and $p \in cl_{\mathscr{J}(A)} K$, it follows that $p \in cl_{\mathscr{J}(A)} F \cap K$ and therefore it is readily seen that $\hat{\sigma}(p)$ is a prime filter.

Once again, we begin by showing that $\mathscr{X}$ is an ultrafilter if and only if $\mathscr{X} = \hat{\sigma}(M)$ where M is a maximal ideal. If M is a maximal ideal and $F \notin \hat{\sigma}(M)$, then $M \notin cl_{\mathscr{J}(A)} F = \bigcap_{a \in kF} H_{\mathscr{J}(A)}(a)$. Thus for some $a \in kF$, $M \notin cl_{\mathscr{J}(A)} H_{\mathscr{M}(A)}(a) = H_{\mathscr{J}(A)}(a)$ and $a \notin M$. Consequently for some $m \in M$ and $b \in A$, $ab + m = e$ and $H_{\mathscr{M}(A)}(a) \cap H_{\mathscr{M}(A)}(m) = \varnothing$. Thus $F \cap H_{\mathscr{M}(A)}(m) = \varnothing$. Of course $\hat{\sigma}(M) \cap L_{\mathscr{M}(A)} = \sigma(M)$ and therefore $H_{\mathscr{M}(A)}(m) \in \hat{\sigma}(M)$. Suppose now that $\mathscr{X}$ is an ultrafilter in $J(\mathscr{M}(A), \mathscr{C})$. Let $p = \{p \in A / H_{\mathscr{M}(A)}(a) \in \mathscr{X}\}$. As $\mathscr{X}$ is an ultrafilter in $\mathscr{C}$, then $\mathscr{X} \cap L_{\mathscr{M}(A)}$ is an ultrafilter in $L_{\mathscr{M}(A)}$ and by part (a), $M = \sigma^{-1}(\mathscr{X} \cap L_{\mathscr{M}(A)})$ is a maximal ideal. By the strong properties of the closure operator in very dense spaces (Prop. 2), as M is in the closure of every set in $\sigma(M) = \mathscr{X} \cap L_{\mathscr{M}(A)}$ by (*a*), and every set in $\mathscr{X}$ is an intersection of sets in $\sigma(M)$, M is in the closure of every set in $\mathscr{X}$ and we are done.

To show that if $p \in \mathscr{J}(A)$, then $\hat{\sigma}(p) \in J(\mathscr{M}(A), \mathscr{C})$, we need only show that $\hat{\sigma}(p) = \bigcap_{\hat{\sigma}(p) \subset \hat{\sigma}(M)} \hat{\sigma}(M)$. But if $F \in \hat{\sigma}(M)$ for all M such that $p \subset M$, $M \in cl_{\mathscr{J}(A)} F$ for all such M and equivalently $kF \subset M$ for all such M. Hence $kF \subset p = \bigcap_{p \subset M} M$ and $F \in \hat{\sigma}(p)$.

To show that $\hat{\sigma}$ is onto, let $\mathscr{J} \in J(\mathscr{M}(A), \mathscr{C})$. Then $\mathscr{J} = \bigcap_{\mathscr{J} \subset \hat{\sigma}(M)} \hat{\sigma}(M)$ and $\mathscr{J} \cap L_{\mathscr{M}(A)} = \bigcap \hat{\sigma}(M) \cap L_{\mathscr{M}(A)} = \bigcap_{\mathscr{J} \cap L_{\mathscr{M}(A)} \subset \sigma(M)} \sigma(M)$ and $\mathscr{J} \cap L_{\mathscr{M}(A)} = \sigma(p)$ for some $p \in \mathscr{J}(A)$. Once again as p is in the closure of each set in $\mathscr{J} \cap L_{\mathscr{M}(A)}$ and every set in $\mathscr{J}$ is an intersection of sets in $\mathscr{J} \cap L_{\mathscr{M}(A)}$, p is in the intersection of each set in $\mathscr{J}$ and it readily follows that $\hat{\sigma}(p) = \mathscr{J}$.

The proofs of the facts that σ is a $1 - 1$ bicontinuous map are straight forward and left to the reader.

PROPOSITION 16. *If* A *is a commutative ring with identity, then there is a* $1 - 1$ *correspondence between the ideals* $p \in \mathscr{J}(A)$ *and the irreducible closed subsets of* $\mathscr{M}(A)$ *where* $p \to (cl_{\mathscr{J}(A)}\{p\}) \cap \mathscr{M}(A)$ *establishes the correspondence.*

Proof. See Prop. 12 and Prop. 15.

Noting that we have now shown that $\mathscr{J}(A)$ is the Jacobson completion of $\mathscr{M}(A)$, as $\mathscr{J}(A)$ is spectral when A is Jacobson, $\mathscr{M}(A)$ is therefore prespectral. We may now state the following proposition.

PROPOSITION 17. *A topological space* X *is the maximal ideals of a Jacobson ring if and only if it is a prespectral* T_1 *space.*

We close the paper with a few examples.

Example 1. If X is a compact Hausdorff space, then $J(X, \mathscr{C}) = W(X, \mathscr{C})$, This follows from Prop. 11 and the fact that every closed subet of X with more than one point is reducible. It follows that a regular semi-simple Banach algebra [2] is not an integral domain, for the ideal consisting of the zero vector would then be a Jacobson prime ideal and this cannot be by Prop. 15.

Example 2. Let X be an infinite set with cofinite topology. Then the following are all true:

(a) X is a prespectral T_1 space;

(b) $W(X, \mathscr{C}) = X$;

(c) $J(X, \mathscr{C}) = X \cup \{\mathscr{L}_X\}$ where $\mathscr{L}_X = \{X\}$.

These three statements lead to the conclusion that any Jacobson ring for which X is the maximal ideals will have a unique prime ideal which is the radical of the ring. This prime ideal will be the zero ideal if and only if the ring is an integral domain. A result of this sort can be found in [2].

Example 3. Let X be a O-dimensional Hausdorff space [1] and F a rank one nontrivially nonarchimedean valued-field of characteristic zero. Let C (X, F) denote the continuous F-valued functions on X and I_x the ideal in C (X , F) generated by all characteristic functions of closed and open (clopen) sets O such that $x \in CO$. The maximal ideal of all functions in C (X , F) which vanish at x will be denoted by M_x. In X, the zero sets of functions of C (X , F) are the C_δ sets (denumerable intersections of clopen sets). See [1] for a proof of this. The following statements are all true.

(a) If X is compact, then $\mathrm{cl}_{C(X,F)} I_x = M_x$.

(b) If p is a prime ideal in C (X , F), then if X is compact, there exists a unique $x \in X$ such that $\mathrm{cl}_{C(X,F)}\, p = M_x$.

(c) C (X , F) is biregular [6] if and only if all C_δ sets in X are clopen. In this case with $A = C(X, F)$, $P(\mathscr{M}(A), L_{\mathscr{M}(A)}) = W(\mathscr{M}(A), L_{\mathscr{M}(A)}) = J(\mathscr{M}(A), L) = \beta_0(X)$ where $\beta_0(X)$ is the Banaschewski compactification of X, and every prime ideal of C (X , F) is maximal.

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