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Vincenzo Malvestuto, Francesco Zirilli

Some effects of the horizontal component of Earth rotation in a stratified inviscid Ocean with constant depth and exponential density

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Oceanografia. — *Some effects of the horizontal component of Earth rotation in a stratified inviscid Ocean with constant depth and exponential density.* Nota di VINCENZO MALVESTUTO e FRANCESCO ZIRILLI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Gli effetti di una stratificazione stabile iniziale di tipo esponenziale e quelli dovuti all'inclinazione sulla verticale dell'asse di rotazione terrestre, studiati separatamente in due precedenti Note lincee [1], [2], vengono qui approfonditi cumulativamente.

Aim of this paper is the generalization of the results appeared in two preceding works—[1] and [2]—to the case of a exponentially stratified ocean with constant depth (incompressible and non viscous) with vertical and horizontal component of Earth's rotation, λ and μ, both non zero.

The complete differential system describing the motion of our ocean is—see [1]—:

$$
\begin{aligned}
u_t &= \lambda v - \mu w - \frac{1}{\rho_0} p_x \\
v_t &= -\lambda u - \frac{1}{\rho_0} p_y \\
w_t &= \mu u - g \frac{1}{\rho_0} \delta - \frac{1}{\rho_0} p_z \\
\delta_t &= -\frac{d\rho_0}{dz} w \\
u_x &+ v_y + w_z = 0
\end{aligned}
\tag{1}
$$

where the symbols have the usual meaning. The initial conditions are homogeneous.

By introducing, as usually, the " velocity " W, defined as follows

$$
W = \frac{1}{\rho_0} \frac{g\delta}{\nu} \qquad \text{with} \quad \nu^2 = -g \frac{1}{\rho_0} \frac{d\rho_0}{dz} \tag{2}
$$

eqq. (1) become more simply:

$$
\begin{aligned}
u_t &= \lambda v - \mu w - \rho_0^{-1} p_x \\
v_t &= -\lambda u - \rho_0^{-1} p_y \\
w_t &= \mu u - \nu W - \rho_0^{-1} p_z \\
W_t &= \nu w \\
u_x &+ v_y + w_z = 0
\end{aligned}
\tag{3}
$$

(*) Nella seduta del 14 dicembre 1974.

from which the semigroup structure brightly appears. It will occur to study the semigroup generated by the bounded operator A:

$$A \equiv \begin{pmatrix} 0 & \lambda & -\mu & 0 \\ -\lambda & 0 & 0 & 0 \\ \mu & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix} \tag{4}$$

Such a semigroup exsists, for A is a bounded operator by virtue of the fact that λ and μ are assumed not dependent on the space-variables, and that ν is also in the general case a bounded well-behaved function of the depth, unless there exist some discontinuous jumps in the ocean's density.

Said $U(t)$ such a semigroup, let us set:

$$U(t) = e^{tA} = ((U_{ij}(t))_{i,j=1,2,3,4}\,. \tag{5}$$

The semigroup $U(t)$ will satisfy the following equations:

$$\begin{aligned} &\frac{dU}{dt} = AU(t) \\ &U(0) = I \qquad \text{(identity operator)}, \end{aligned} \tag{6}$$

that means for the U_{ij}'s to verify the differential equations:

$$\begin{aligned} \dot U_{1j} &= \lambda U_{2j} - \mu U_{3j} \\ \dot U_{2j} &= -\lambda U_{1j} \\ \dot U_{3j} &= \mu U_{1j} - \nu U_{4j} \\ \dot U_{4j} &= \nu U_{3j} \end{aligned} \qquad j = 1, 2, 3, 4, \tag{7}$$

with the initial conditions:

$$U_{ij}(0) = \delta_{ij}\,. \tag{8}$$

Deriving with respect to the time the first and the fourth of eqq. (7) and use making of the other two, we obtain for the two unknown $U_{1j}(t)$ and $U_{4j}(t)$ the following coupledoscillator system of equations:

$$\begin{aligned} \ddot U_{1j} &= -\sigma^2 U_{1j} + \mu\nu U_{4j} \\ \ddot U_{4j} &= \mu\nu U_{1j} - \nu^2 U_{4j} \end{aligned} \qquad \sigma \equiv \lambda^2 + \mu^2. \tag{9}$$

The characteristic of the semigroup are found solving the related secular equation, connected with the matrix:

$$\begin{pmatrix} \sigma^2 & -\mu\nu \\ -\mu\nu & \nu^2 \end{pmatrix} \tag{10}$$

that is

$$K^4 - (\rho^2 + \nu^2) K^2 + \nu^2 \lambda^2 = 0 . \tag{11}$$

The positive root of (11) are by definition the squares of the required fundamental frequencies:

$$K_{1,2}^2 = \frac{1}{2} \{ (\sigma^2 + \nu^2) \pm \sqrt{(\sigma^2 + \nu^2)^2 - (2 \nu \lambda)^2} \} , \tag{12}$$

with both roots positive:

$$K_{1,2} = \frac{1}{2} \{ \sqrt{(\lambda + \nu)^2 + \mu^2} \pm \sqrt{(\lambda - \nu)^2 + \mu^2} \} \tag{12'}$$

From eq. (12) it is possible to see that, if were $\mu = 0$, we would obtain as fundamental frequencies just those found in a preceding paper—see [1]—; more precisely, λ, proper of the horizontal motion of our ocean, and ν, typical of vertical motions connected with the w-velocity waves and the W-density waves. This fact underlines the interesting meaning of the μ component as a mixing parameter which breaks the degeneracy of the problem of the motion, coupling horizontal and vertical waves and shifting the fundamental frequencies of the semigroup.

Let us now go on in computing the matrix's terms $U_{ij}(t)$. Their structure is really already known, for they must result from linear combinations buit up with the four indipendent eigenfunctions:

$$\sin K_1 t \quad , \quad \cos K_1 t \quad , \quad \sin K_2 t \quad , \quad \cos K_2 t .$$

Therefore we start on writing:

$$U(t) = B_1 \sin K_1 t + C_1 \cos K_1 t + B_2 \sin K_2 t + C_2 \cos K_2 t$$

and determine the unknown time-indipendent matrix B_1, C_1, B_2 and C_2 using the remarkable relation:

$$\left. \frac{d^s U}{dt^s} \right|_{t=0} = A^s \qquad (s = 0, 1, 2, \cdots)$$

for the first four values of $s : s = 1, 2, 3$.

A lenghty computation yields at last:

$$e^{tA} = \left(\frac{1}{K_1^2} - \frac{1}{K_2^2} \right)^{-1} \left[I \left(\frac{\cos K_1 t}{K_1^2} - \frac{\cos K_2 t}{K_2^2} \right) + A \left(\frac{\sin K_1 t}{K_1^3} - \frac{\sin K_2 t}{K_2^3} \right) \right] -$$

$$- (K_1^2 - K_2^2)^{-1} \left[A^2 (\cos K_1 t - \cos K_2 t) + A^3 \left(\frac{\sin K_1 t}{K_1} - \frac{\sin K_2 t}{K_2} \right) \right] ,$$

that is in explicit form:

$$((\mathrm{U}_{ij})) = \frac{1}{\Delta^{1/2}} \begin{pmatrix} -\langle(\nu^2 - \mathrm{K}_i^2)\cos \mathrm{K}_i t\rangle & -\lambda\left\langle(\nu^2-\mathrm{K}_i^2)\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle & -\mu\langle \mathrm{K}_i \sin \mathrm{K}_i t\rangle & -\mu\nu\langle\cos \mathrm{K}_i t\rangle \\ \lambda\left\langle(\nu^2-\mathrm{K}_i^2)\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle & \lambda^2\left\langle\left(1-\frac{\nu^2}{\mathrm{K}_i^2}\right)\cos \mathrm{K}_i t\right\rangle & -\mu\lambda\langle\cos \mathrm{K}_i t\rangle & \lambda\mu\nu\left\langle\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle \\ \mu\langle \mathrm{K}_i \sin \mathrm{K}_i t\rangle & -\mu\lambda\langle\cos \mathrm{K}_i t\rangle & -\langle(\lambda^2-\mathrm{K}_i^2)\cos \mathrm{K}_i t\rangle & \nu\left\langle(\lambda^2-\mathrm{K}_i^2)\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle \\ -\mu\nu\langle\cos \mathrm{K}_i t\rangle & -\mu\nu\lambda\left\langle\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle & -\nu\left\langle(\lambda^2-\mathrm{K}_i^2)\frac{\sin \mathrm{K}_i t}{\mathrm{K}_i}\right\rangle & \nu^2\left\langle\left(1-\frac{\lambda^2}{\mathrm{K}_i^2}\right)\cos \mathrm{K}_i t\right\rangle \end{pmatrix}$$

where K_1^2 and K_2^2 are the roots of secular equation (12) and

$$\Delta \equiv (\mathrm{K}_1^2 - \mathrm{K}_2^2)^2$$

is the discriminant of such equation.

Furthermore we have adopted the following notation for the sake of brevity:

$$\langle f(i)\rangle = f(1) - f(2)$$

that is $\langle\ \rangle$ means complete antisymmetrization.

Before going on, let us make a brief remark about the significance of the frequency ν defined in (2). From a glance at es. (12) it is immediate to recognize that it is just the presence of a stratification that introduces into the motions a frequency $\nu(z)$, generally depending upon the depth, which influences the formation of internal waves. If were $\nu = 0$, that is density uniform, we would get again from eqq. (12) and (12′) as a particular case the unique frequence σ, which influences both vertical and horizontal motions in a homogeneous ocean—see [2]—.

In the remaining, starting from the knowledge of the explicit form of the semigroup $\mathrm{U}(t)$, we will retrace the steps made in the two preceding papers— [1] and [2]—; so we shall arrive at an equation for the "pressure" $(p(x, y, z, t)$, without passing through the heavy task of evaluating the entire tridimensional velocity field $\boldsymbol{u} = (u, v, w)$. It is just and merely at this stage of computation that we introduce the hypothesis of an exponential density, that is

$$\rho_0(z) = ce^{-\alpha z} \qquad \text{or} \quad \nu(z) = \text{const.} = [g\alpha]^{1/2}.$$

After applying the generalized Duhamel principle, substituting in the continuity equation, deriving under the integral signs with respect to the space-variables and performing the Laplace transform with respect to the time t, we obtain the following version of the continuity equation (P denotes the Laplace transform of $p(x, y, z, t)$):

$$\left\langle \frac{K_i^2 - \nu^2}{K_i^2 + \omega^2} \right\rangle P_{xx} + \left\langle \frac{\lambda^2}{K_i^2} \frac{K_i^2 - \nu^2}{K_i^2 + \omega^2} \right\rangle P_{yy} + \tag{13}$$

$$+ \left\langle \frac{K_i^2 - \lambda^2}{K_i^2 + \omega^2} \right\rangle P_{zz} + 2 \left\langle \frac{\mu\lambda}{K_i^2 + \omega^2} \right\rangle P_{yz} +$$

$$+ \alpha \left[\frac{\mu}{\omega} \left\langle \frac{K_i^2}{K_i^2 + \omega^2} \right\rangle P_x - \left\langle \frac{\mu\lambda}{K_i^2 + \omega^2} \right\rangle P_y + \left\langle \frac{K_i^2 - \lambda^2}{K_i^2 + \omega^2} \right\rangle P_z \right] = 0.$$

The boundary conditions for eq. (13), derived in the hydrostatic approximation as in ref. [1] are:

$$\begin{cases} P(x, y, 0, \omega) = f(x, y, \omega) \\ \dfrac{\partial P}{\partial z}(x, y, -h, \omega) = 0 \end{cases} \quad h \text{ is the constant depth,} \tag{14}$$

where f is a given function and the ocean is considered as extended to infinity in the x– and y–directions.

We will look for simplicity at solutions indipendent of y, and in order to use the separated variables method, we set:

$$f(x, \omega) = A(\omega)\, e^{-b\frac{\alpha}{2}x} \cos Kx \tag{15}$$

with b and K constant to be determined; the choice (15) is more general than that made in [1]; in fact, the presence of the horizontal component of Earth's rotation μ may introduce into the horizontal surface waves a damping factor depending on α. Actually we will find $b \neq 0$. The solution of (13), (14) with the position (15) is:

$$P(x, z, \omega) = A(\omega)\, e^{-\frac{\alpha}{2}(bx+z)} \cos Kx \cosh(\gamma z)\,[1 + \mathrm{tgh}(\gamma h + \beta)\,\mathrm{tgh}(\gamma z)], \tag{16}$$

where:

$$\gamma^2 = \frac{\alpha^2}{4}\left[1 + \frac{\mu^2\omega^2}{(\omega^2 + \lambda^2)(\omega^2 + g\alpha)}\right] + K^2 \left(\frac{\omega^2 + g\alpha}{\omega^2 + \lambda^2}\right),$$

$$b = \mu\left(\frac{\omega}{\omega^2 + \nu^2}\right),$$

$$\beta = \text{sitt sinh}\left\{\frac{\alpha}{2K}\left[\frac{(\omega^2 + \lambda^2)(\omega^2 + g\alpha)}{\mu^2\omega^2\left(\frac{\alpha}{2K}\right)^2 + (\omega^2 + g\alpha)^2}\right]^{1/2}\right\},$$

having assumed the wavenumber K fixed by external conditions.

As usual the poles of eq. (16), satisfying the well known relation:

$$\operatorname{tgh}(\gamma h) = -\frac{\gamma}{\left(\frac{\alpha}{2}\right)},$$

yield the allowed characteristic frequencies.

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[2] V. MALVESTUTO and F. ZIRILLI (1974) – *Inertial waves in a homogeneous ocean. The dispersion relation for standing waves in presence of the horizontal component of Earth's rotation*, « Rend. Acc. Naz. Lincei », *57*, 418–421.