Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Description of a class of differential equations with set-valued solutions. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.2, p. 158–162.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_2_158_0>

Equazioni funzionali. — *Description of a class of differential equations with set-valued solutions*. Nota I di MICHAŁ KISIELEWICZ, presentata (*) dal Socio G. CIMMINO.

RIASSUNTO. — Nelle presenti Note (I e II) proviamo il teorema di tipo Orlicz per equazioni differenziali con soluzioni a valori che sono insiemi compatti convessi. Questa Nota contiene le definizioni di base e la dimostrazione della completezza di uno spazio metrico fondamentale.

INTRODUCTION

In [6] W. Orlicz proved that the class of all differential equations of the form $y' = f(t, y)$ which have more than one solution is of the Baire first category. Later on, this type of theorem was proved for a partial differential equation of the hyperbolic type by A. Alexiewicz and W. Orlicz ([1]). The aim of the present Notes (I and II) is to give a proof of the analogous theorem for differential equations with set-valued solutions of the form:

$$\text{(1)} \qquad DX = F(t, X),$$

where $F : [0, T] \times H \to H$ is a given mapping and (H, r) denotes the metric space of all nonempty compact convex subsets of the Euclidean space R^n with the metric function r given by the Hausdorff distance. It is known ([4]) that (H, r) is a complete metric space. The existence and uniqueness of solutions of the initial value problem of (1) have been proved in [2] and [3]. In § 1 we give some basic definitions and conventions. The § 2 contains the proof of the completeness of a fundamental metric space which we introduce in this paragraph.

§ 1. BASIC DEFINITIONS AND CONVENTIONS

Let R^n denote the Euclidean n-space and denote by (H, r) the metric space of all non-empty compact convex subsets of R^n, where r is the metric given by the Hausdorff distance. If A and B are given points of H, it is defined $A + B = \{x + y : x \in A, y \in B\}$ and $\lambda \cdot A = \{\lambda x : x \in A\}$ where $\lambda \in R_1$ and $\lambda \geq 0$. We define the difference $A - B$ as the set C, if it exists, such that $A = B + C$. In [2] was proved the following Lemma:

LEMMA 1. *Let* $\lambda \geq 0$, $X, Y, U, V \in H$ *and suppose differences* $X - U$ *and* $Y - V$ *exist. Then*

$$r(X + U, Y + V) \leq r(X, Y) + r(U, V),$$
$$r(X - U, Y - V) \leq r(X, Y) + r(U, V),$$
$$r(X + U, Y + U) = r(X, Y),$$
$$r(\lambda \cdot X, \lambda \cdot Y) = \lambda r(X, Y).$$

(*) Nella seduta dell'8 febbraio 1975.

Let D be a measurable subset of R^1 such that $\mu(D) \in (0, \infty)$. A function $F : D \to H$ is said to be measurable if for each $C \in H$ the set $\{t : F(t) \cap C = \varnothing\}$ is Lebesgue measurable. The mapping F is called Hukuhara integrable if the single-valued function $\| F(t) \| = r(F(t), 0)$, where $0 = (0, \cdots, 0)$, is Lebesgue integrable on D. In this case we shall denote by $\int_D F(t)\, dt$ the Hukuhara integral of F on D. The mapping $F : H \to H$ is called continuous in $C \in H$ if for every number $\eta > 0$ there is a number $\delta > 0$ such that for $X \in H$ such that $r(C, X) < \delta$ we have $r(F(C), F(X)) < \eta$.

Let $X : [\alpha, \beta] \to B$ be a given mapping. Using the definition of the difference in H, the Hukuhara derivative ([5]) of X may be introduced in the following way:

$$(2) \qquad DX(t) = \lim_{h \to 0^+} (1/h) \cdot (X(t+h) - X(t)) = \lim_{h \to 0^+} (1/h) \cdot (X(t) - X(t-h))$$

where X is assumed to belong to the class $\mathscr{D}$ (clearly not empty) of all functions such that the differences in (2) are possible. The mapping X is called Hukuhara differentiable in $[\alpha, \beta]$ if $DX(t)$ exists for every $t \in [\alpha, \beta]$.

We recall some topological notations in (H, r). Let $\{A_n\}$ be a sequence of (H, r). The sequence $\{A_n\}$ is said to be convergent to $A \in H$ if $\lim_{n \to \infty} r(A_n, A) = 0$. Let $S \subset H$ and let $\bar{S}$ denote the closure of S. We shall write $A \in \bar{S}$ if and only if there is a sequence $\{A_n\}$ such that $A_n \in S$ for $n = 1, 2, \cdots$ and $r(A_n, A) \to 0$ as $n \to \infty$. We call the set $S \subset H$ dense in $B \subset H$ if $B \subset \bar{S}$. We shall call the set $S \subset H$ non-dense if there is not a ball K of (H, r) such that $K \subset \bar{S}$. The set $S \subset H$ is said to be of the Baire's first category in (H, r) if there exists a sequence $\{S_n\}$ of non-dense subsets of (H, r) such that $S = \bigcup_{n=1}^{\infty} S_n$.

Finally, we recall the Arzelà theorem for multi-valued mappings ([4]).

THEOREM (Arzelà). *Suppose $\{X_n(t)\}$ is a sequence of mappings from $[\alpha, \beta]$ to H which is equicontinuous and uniformly bounded on $[\alpha, \beta]$. Then there is an uniformly converging subsequence of $\{X_n(t)\}$.*

§ 2. THE FUNDAMENTAL METRIC SPACE

Suppose $F : [0, T] \times H \to H$ is a mapping such that the following Hypotheses H(F) are fulfilled. *Hypotheses* H(F):

(i) $F(\cdot, X)$ is measurable for every fixed $X \in H$,

(ii) $F(t, \cdot)$ is continuous for every fixed $t \in [0, T]$,

(iii) there exists a Lebesgue integrable function $\varphi : [0, T] \to R^1$ such that $\| F(t, X) \| \leq \varphi(t)$ for every $(t, X) \in [0, T] \times H$, where $\| F(t, X) \| = r(F(t, X), 0)$; $0 = (0, \cdots, 0)$,

(iv) for every $\eta > 0$ there exists a mapping $G_\eta : [0, T] \times H \to H$ such that

(a) G_η satisfies (i)-(iii);

(b) G_η is uniformly Lipschitz continuous with respect to X, i.e. there is a number $L > 0$ such that $r(G_\eta(t, X), G_\eta(t, Y)) \leq L r(X, Y)$ for $X, Y \in H$ and every $t \in [0, T]$;

(c) $r(F(t, X), G_\eta(t, X)) < \eta$ for every $(t, X) \in [0, T] \times H$.

Let $\mathscr{B}$ denote the class of all mappings F satisfying the Hypotheses H (F). An equivalence relation $\sim$ is defined on $\mathscr{B}$ by stating that $F_1 \sim F_2$ if $F_1(t, X) = F_2(t, X)$ for almost every $t \in [0, T]$ and fixed $X \in H$. The equivalence class containing F is denoted by $\tilde{F}$. The space $\mathscr{F}$ is taken to be the quotient space $\mathscr{B}/\sim$. A metric $\rho_{\mathscr{F}}$ on $\mathscr{F}$ is defined by

$$\rho_{\mathscr{F}}(\tilde{F}_1, \tilde{F}_2) = \int_0^T \sup_{X \in H} r(F_1(t, X), F_2(t, X))\, dt \quad \text{for} \quad F_1 \in \tilde{F}_1, \quad F_2 \in \tilde{F}_2. \tag{3}$$

We shall prove the following Theorem:

THEOREM 1. *$(\mathscr{F}, \rho_{\mathscr{F}})$ is a complete metric space.*

Proof. Let $\{\tilde{F}_n\}$ be a sequence of $\mathscr{F}$ such that $\rho_{\mathscr{F}}(\tilde{F}_n, \tilde{F}_m) \to 0$ as $n, m \to \infty$, and let $F_n \in \tilde{F}_n$, $F_m \in \tilde{F}_m$. For every $\eta > 0$ there is $N = N(\eta)$ such that

$$\int_0^T \sup_{X \in H} r(F_n(t, X), F_m(t, X))\, dt < \eta$$

for $n, m \geq N(\eta)$. Suppose $\{n_k\}$ to be such that $n_1 < n_2 < \cdots$ and $n_k \geq N(1/2^{2k})$. Then

$$\sup_{X \in H} r(F_{n_k}(t, X), F_{n_{k-1}}(t, X))\, dt \leq 1/2^{2k}$$

for $k = 1, 2, \cdots$. Taking $A_k = \{t : \sup_{X \in H} r(F_{n_k}(t, X), F_{n_{k-1}}(t, X)) > 1/2^k\}$ we have

$$1/2^{2k} \geq \int_{A_k} \sup_{X \in H} r(F_{n_k}(t, X), F_{n_{k-1}}(t, X))\, dt \geq (1/2^k) \cdot \mu(A_k).$$

Then $\mu(A_k) \leq 1/2^k$. Let $A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$. Since $\mu(A) \leq \mu\left(\bigcup_{k=i}^{\infty} A_k\right) \leq \sum_{k=i}^{\infty} \mu(A_k) < \sum_{k=i}^{\infty} (1/2^k) = 1/2^{i-1}$ for $i = 1, 2, \cdots$, then $\mu(A) = 0$. Let $A^\sim = [0, T] \setminus A$ and $A_k^\sim = [0, T] \setminus A_k$. We have $A^\sim = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} A_k^\sim$. Then for $t \in A^\sim$ there is a number i such that for every $k \geq i$ we have $\sup_{X \in H} r(F_{n_k}(t, X), F_{n_{k-1}}(t, X)) \leq \leq 1/2^k$. Since for an arbitrary $k, m \geq i$ such that $k < m$ we have

$$\sup_{X \in H} r(F_{n_k}(t, X), F_{n_m}(t, X)) \leq$$

$$\leq \sup_{X \in H} r(F_{n_k}(t, X), F_{n_{k+1}}(t, X)) + \cdots + \sup_{X \in H} r(F_{n_{m-1}}(t, X), F_{n_m}(t, X)) \leq$$

$$\leq (1/2)^{k+1} - (1/2)^{m+1}$$

then $\sup_{X \in H} r(F_{n_k}(t, X), F_{n_m}(t, X)) \to 0$ as $k, m \to \infty$ for $t \in A^\sim$. The space (H, r) is a complete metric space, then for every fixed $(t, X) \in A^\sim \times H$ the sequence $\{F_{n_k}(t, X)\}$ is convergent to some element $G(t, X) \in H$. Therefore we have the mapping $G : A^\sim \times H \to H$ such that $r(F_{n_k}(t, X), G(t, X)) \to 0$ as $k \to \infty$ for every $(t, X) \in A^\sim \times H$. The function G is measurable in t for every fixed $X \in H$. We shall show that $\sup_{X \in H} r(F_{n_k}(t, X), G(t, X)) \rightrightarrows 0$ as $k \to \infty$ for $t \in A^\sim$. Indeed, let $g_k(t, X) = r(F_{n_k}(t, X), G(t, X))$ for $(t, X) \in A^\sim \times H$. For every $k = 1, 2, \cdots$ and $(t, X) \in A^\sim \times H$ we have $|g_k(t, X) - g_{k-1}(t, X)| \leq r(F_{n_k}(t, X), F_{n_{k-1}}(t, X))$. Therefore $\sup_{X \in H} |g_k(t, X) - g_{k-1}(t, X)| \leq 1/2^k$ for every $t \in A^\sim$ and $k \geq i$. Hence it is easy to see that the series $g_0(t, X) + \sum_{k=1}^{\infty} [g_k(t, X) - g_{k-1}(t, X)]$ is absolutely and uniformly convergent. Consequently, the sequence $\{g_k(t, X)\}$ is uniformly convergent on $A^\sim \times H$. Therefore for $t \in A^\sim$ we have $\sup_{X \in H} r(F_{n_k}(t, X), G(t, X)) \rightrightarrows 0$ as $k \to \infty$. Let $F : [0, T] \times H \to H$ be the mapping defined by

$$F(t, X) = \begin{cases} G(t, X) & \text{for } (t, X) \in A^\sim \times H \\ \{0\} & \text{for } (t, X) \in A \times H . \end{cases}$$

The mapping F is measurable in t for fixed $X \in H$. It is continuous in X for every fixed $t \in A$. Furthermore for any $C \in H$ and arbitrary $\eta > 0$ there exists a number $\delta > 0$ such that $r(F_{n_k}(t, X), F_{n_k}(t, C)) < \eta/3$ whenever $r(X, C) < \delta$ for $k = 1, 2, \cdots$ and $t \in [0, T]$. Hence and from the uniform convergence of $\{F_{n_k}(t, X)\}$ it follows that F is continuous in X for fixed $t \in A^\sim$. It is easy to see that $\| F(t, X) \| \leq \varphi(t)$ for $(t, X) \in [0, T] \times H$. Now, suppose N is such that $r(F_N(t, X), F(t, X)) < \eta/2$ for $(t, X) \in A^\sim \times H$ and let $G_{\eta/2}$ satisfy the conditions (*a*), (*b*) of (iv) and suppose that $r(F_N(t, X), G_{\eta/2}(t, X)) < \eta/2$ for $(t, X) \in A^\sim \times H$. Taking

$$G_\eta(t, X) = \begin{cases} G_{\eta/2}(t, X) & \text{for } (t, X) \in A^\sim \times H \\ \{0\} & \text{for } (t, X) \in A \times H \end{cases}$$

we have $r(F(t, X), G_\eta(t, X)) < \eta$ for every $(t, X) \in [0, T] \times H$. Therefore

$F \in \mathscr{B}$ and $\tilde{\tilde{F}} \in \mathscr{F}$. We shall show that $\rho_{\mathscr{F}}(\tilde{\tilde{F}}_n, \tilde{\tilde{F}}) \to 0$ as $n \to \infty$. For $n, k \geq N(\eta)$ we have

$$\int_0^T \sup_{X \in H} r(F_n(t, X), F_{n_k}(t, X))\, dt \leq \eta .$$

Taking for fixed $n \geq N(\eta)$

$$\Phi_k(t) = \sup_{X \in H} r(F_n(t, X), F_{n_k}(t, X))$$

in virtue of Fatou's Lemma we have

$$\int_0^T \underline{\lim}_{k \to \infty} \Phi_k(t)\, dt \leq \underline{\lim}_{k \to \infty} \int_0^T \Phi_k(t)\, dt = \underline{\lim}_{k \to \infty} \rho_{\mathscr{F}}(\tilde{\tilde{F}}_n, \tilde{\tilde{F}}_{n_k}) \leq \eta$$

for $n \geq N(\eta)$. Let us observe that $\lim_{k \to \infty} \sup_{X \in H} r(F_{n_k}(t, X), F(t, X)) = 0$ for $t \in A^{\sim}$ implies $\lim_{k \to \infty} \rho_{\mathscr{F}}(\tilde{\tilde{F}}_{n_k}, \tilde{\tilde{F}}) = 0$ ([5]). Therefore for $n \geq N(\eta)$ we have $\rho_{\mathscr{F}}(\tilde{\tilde{F}}_n, \tilde{\tilde{F}}) \leq \eta$. This completes the proof.

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