Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Linear operators on certain completions of the s-d-ring over the integers. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.3, p. 271–276.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_3_271_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'8 marzo 1975

Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Linear operators on certain completions of the s-d-ring over the integers.* Nota I di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Questa Nota I determina le varie possibilità di automorfismi, semi–derivazioni e semi–integrazioni nell' s–d–anello sopra gli interi, e costituisce una preparazione per una successiva Nota II.

INTRODUCTION

The following paper studies linear operators on certain s–d–rings. Special attention is devoted to semi-derivations, semi-integrations and natural generalizations of these operators. They are intimately connected with other linear operators, namely algebra automorphisms and certain algebra homomorphisms.

Part I (Nota I) deals with the study of linear operators on the s–d–ring $\mathfrak{A}$ over the ring of integers **Z**. All **Z**-algebra homomorphisms from $\mathfrak{A}$ into **Z** and all **Z**-algebra automorphisms of $\mathfrak{A}$ are first enumerated. From these, we are then able to construct all possible semi-derivations and semi-integrations on $\mathfrak{A}$. Their mutual interdependence is obtained by the process of conjugation.

It is remarkable that one is able to find to each semi-integration S on $\mathfrak{A}$ a completion $\hat{\mathfrak{A}}$ to which S may be extended and on which S has an inverse S^{-1}. This inverse is almost a semi-derivation, indeed it satisfies the product rule.

Linear operators on $\hat{\mathfrak{A}}$ are studied in Part II (Nota II). We show there that an automorphism of $\mathfrak{A}$ extends always to a certain homomorphism on $\hat{\mathfrak{A}}$, which however, in the nontrivial case, is never an automorphism on $\hat{\mathfrak{A}}$.

(*) Nella seduta dell'8 febbraio 1975.

With help of these extensions, we can construct semi-derivations and then semi-integrations on $\hat{\mathfrak{A}}$. The ring of constants, however, will in general be a subring $\mathbf{Z}_m$ if $\hat{\mathfrak{A}}$, which is larger than $\mathbf{Z}$, so that $\hat{\mathfrak{A}}$ must be regarded as a $\mathbf{Z}_m$-algebra.

In another paper, we will show that an *analysis* may be developed on $\hat{\mathfrak{A}}$, which is much richer than expected. We have already made use of it in the present paper, when we determined the kernels of the above mentioned homorphisms and semi-derivations on $\hat{\mathfrak{A}}$ by solving certain differential equations. The element $x_{-1} = S^{-1}(1)$ plays two roles in our analysis. On one hand it takes over the role of the Dirac δ-function which is important in ordinary analysis. On the other hand, there exists a semi-derivation $\bar{D}$ on $\hat{\mathfrak{A}}$ for which $\bar{D}x_{-1} = x_{-1}$, so that x_{-1} has the differential equation of the exponential function with respect to $\bar{D}$. Algebraically, however, x_{-1} does not act like the exponential function at all. The product of x_{-1} with any element of $\hat{\mathfrak{A}}$ is again x_{-1}, which shows—by the way—that $\hat{\mathfrak{A}}$ is not an integral domain.

Part I

Let $\mathfrak{A}$ be the S–D–ring over the integers $\mathbf{Z}$ (see [1]). We study first the linear operators on $\mathfrak{A}$. Since $\mathfrak{A}$ is an algebra these operators form also an algebra $\mathfrak{L}$. Operators which will be of special interest to us are the two operators occurring in the definition of $\mathfrak{A}$, namely the semi-derivation D which must satisfy the following 5 conditions: (1) Linearity (2) D is onto (3) Product formula: $D(ab) = aDb + bDa - DaDb$ (4) $D(1) = 0$ and (5) $D^{(m)} a = 0$ for some $m < \infty$ for all $a \in \mathfrak{A}$ and the algebra homomorphism σ. From these two operators we constructed the linear operator S, which we called semi-integration, by defining: $S(a) = a' - \sigma(a')$ where a' is any element of $\mathfrak{A}$ such that $Da' = a$. S satisfies the following properties: (6) $DS = I$ (identity) and (7) $SD = I - \sigma$. Other operators which we used before are the algebra homomorphism τ introduced in [1] and Giebutowski used in his thesis [2] the automorphism $I - D$ which turned out to be very helpful for studying p-adic completions of $\mathfrak{A}$. We also need the algebra basis $\{x_i = S^{(i)}(1)\}$, $i = 0, 1, 2, 3, \cdots$. It is clear that L is uniquely determined if we know $L(x_i)$. For example for σ we have

$$\sigma(x_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i > 0, \end{cases}$$

for D we have $D(x_i) = x_{i-1}$ and for S we have $S(x_i) = x_{i+1}$. A linear operator H which is also an algebra homomorphism is already determined by knowing $H(x_1)$ because:

$$H(x_1 x_{i-1}) = H[ix_i - (i-1)x_{i-1}] = iH(x_i) - (i-1)H(x_{i-1})$$
$$= H(x_1)H(x_{i-1}) \qquad \text{so that}$$

$$\text{(I)} \qquad iH(x_i) = [i - 1 + H(x_1)]H(x_{i-1})$$

provides a recursion formula for H (x_i). It is easy to prove that H (x_1) may be arbitrarily prescribed. In this paper we will need this only for two important subcases:

(1) Let σ_m be the algebra homomorphism determined by $\sigma_m(x_1) = -m$. In this case formula (I) provides at once $\sigma_m(x_i) = \binom{i-m-1}{i}$. As special cases we have: σ_0 is the—a priori given-operator σ and σ_{-1} is the operator τ mentioned above. (2) We compute the automorphism group $\mathbf{A}$ of $\mathfrak{A}$. Let $T \in \mathbf{A}$. Suppose deg $T(x_1) = r$. From (I) it follows that deg $T(x_i) = i \cdot r$ and if $a \in \mathfrak{A}$ is of degree t it follows that deg $T(a) = t \cdot r$. Since there exists $a \in \mathfrak{A}$ such that $T(a) = x_1$ it follows that $r \cdot t = 1$ and therefore $r = t = 1$. Let $T(x_1) = \beta_0 + \beta_1 x_1$ and $a = \alpha_0 + \alpha_1 x_1$ then $T(a) = (\alpha_0 + \alpha_1 \beta_0) + \alpha_1 \beta_1 x_1 = x_1$ and we must have: $\alpha_1 = \beta_1 = 1$ or $\alpha_1 = \beta_1 = -1$. In the first case $\alpha_0 = -\beta_0$ and in the second case $\alpha_0 = \beta_0$. There are, therefore, exactly *two* one-parametric families of automorphisms possible which we denote by K^m and ${}^mJ$ respectively. They are determined by:

(II) $\quad K^m(x_1) = m + x_1$ so that by (I) $K^m(x_i) = \sum_{k=0}^{i} \binom{m+i-1-k}{i-k} x_k$

(III) $\quad {}^mJ(x_1) = m - x_1$ so that by (I) ${}^mJ(x_i) = \sum_{k=0}^{i} (-1)^k \binom{m+i-1}{i-k} x_k$.

The automorphisms K^m form an infinite cyclic subgroup $\mathbf{K}$ of $\mathbf{A}$ with index 2. Clearly $K = K^1$ is a generator and $(K)^m = K^m$. $\mathbf{J} = \{ {}^mJ \}$ is the coset of $\mathbf{K}$. From the definition it follows at once that ${}^mJ = {}^0JK^m$ and these automorphisms are involutions: $({}^mJ)^2 = I$. If we put ${}^0J = J$ we can also say that $\mathbf{A}$ is generated by the two elements K and J with the relations: $J^2 = I$ and $JK = K^{-1}J$.

Let us return to the algebra of linear operators $\mathfrak{L}$. On $\mathfrak{L}$ we have the similarity transformations

$$s_A : \mathfrak{L} \to \mathfrak{L}$$
$$L \to L_A = ALA^{-1} \qquad \text{with} \quad A \in \mathbf{A}.$$

If $A = K^m$ we denote L_A also by L_m (m-translations) and if $A = JK^m$ we denote L_A also by L_m^-. In particular $L_0 = L$ and $L^- = JLJ$ (reflection). Note that: $L^{--} = L$ and $(L_m)^- = L_m^- = (L^-)_{-m}$. Let $S_A = \{ s_A \}$. It is the group of similarities on $\mathfrak{L}$ with the multiplication: $s_A \cdot s_B = s_{AB}$. There is a natural group homomorphism from $\mathbf{A}$ onto S_A the kernel of which consists of all $A \in \mathbf{A}$ which commute with all $L \in \mathfrak{L}$. For $A = K^m$ we have $K^mJ = JK^{-m} \neq JK^m$ unless $m = 0$ and for $A = JK^m$ we have $JK^m \cdot J = K^{-m} \neq J \cdot JK^m = K^m$. It follows that $\mathbf{A} \approx S_A$.

Furthermore we need on $\mathfrak{L}$ the mapping $L' = I - L$. Note that: $L'' = L$ and $(L_A)' = (L')_A$.

Let us compute the conjugate classes for some important linear operators:

(1) $L = K^m$. In that case: $L_n = K^n K^m K^{-n} = L$ and $L_n^- = JK^mJ = K^{-m}$.

The conjugate class of K^m consists therefore of two elements only: $\{K^m, K^{-m}\}$ for $m \neq 0$.

(2) $L = JK^m$. In that case: $L_n = K^n JK^m K^{-n} = JK^{m-2n}$ and $L_n^- = (L_n)^- = JJK^{m-2n}J = JK^{-m+2n}$.

The conjugate class of JK^m consists therefore of infinitely many elements: $\{JK^s\}_{s \equiv m \bmod 2}$.

(3) $L = \sigma$. In this case, it is first clear, that L_n and L_n^- are also algebra homomorphisms. Since $L_n(x_1) = K^n \sigma K^{-n}(x_1) = K^n \sigma(-n + x_1) = K^n(-n) = -n$ so that $L_n = \sigma_n$ and $L_n^-(x_1) = J\sigma_n J(x_1) = n$ so that $L_n^- = \sigma_{-n}$.

The conjugate class of σ consists therefore exactly of all algebra homomorphisms σ_n.

(4) $L = D$. In this case, we have first by Giebutowski [2] $D = (K^{-1})'$ so that $D_n = (K^{-1})' = D$ and $D_n^- = D^-$. Since $D(x_1) = 1$ and $D^-(x_1) = JDJ(x_1) = -1$ it follows that $D \neq D^-$. One checks easily that all five conditions for a semi-derivation are satisfied by the operator D^-, so that D^- is another semi-derivation on $\mathfrak{A}$.

The conjugate class of D consists of the two semi-derivations $\{D, D^-\}$.

Now let T be an arbitrary semi-derivation on $\mathfrak{A}$. $T'(ab) = ab - aT(b) - bT(a) + T(a)T(b) = (a - T(a))(b - T(b)) = T'(a) \cdot T'(b)$. It follows that T' is an algebra homomorphism on $\mathfrak{A}$. Note that $T(\alpha) = \alpha \cdot T(1) = 0$ for $\alpha \in \mathbf{Z}$. We have shown before that if $\deg T'(x_1) = n$ then $\deg T'(x_i) = n \cdot i$ and therefore $\deg T(x_i) = i \cdot n$ if $n > 1$. It follows that if m increases deg $T^{(m)}(x_1)$ increases too and condition 5 for a semi-derivative could not hold. If $T(x_1) = \alpha_0 + \alpha_1 x_1$ then obviously $\deg T^{(m)}(x_1) = 1$ for all m and again condition 5 is violated. We must therefore have that $T(x_1) = \alpha \in \mathbf{Z}$.

Now by condition 2 there exists $b \in \mathfrak{A}$ such that $T(b) = 1$. Deg $b = 1$ otherwise $\deg T(b) \neq 0$. Let $b = \beta_0 + \beta_1 x_1$. Since $T(b) = \beta_1 \cdot \alpha = 1$ it follows that $\beta_1 = \alpha = 1$ or $\beta_1 = \alpha = -1$ so that $T'(x_1) = \pm 1 + x_1$. Conclusion: $T' = K$ or K^{-1} which means $T = D^-$ or D. We have therefore the following

THEOREM 1. *The only possible semi-derivations on $\mathfrak{A}$ are the semi-derivations* D *and* D^- *and they are conjugates*: $D^- = JDJ$.

Let $x_{mi} = K^m x_i$ and $x_{mi}^- = JK^m x_i = Jx_{mi}$. It is clear that $\{x_{mi}\}_{m \text{ fixed}}$ and $\{x_{mi}^-\}_{m \text{ fixed}}$ form two families of new algebra basis for $\mathfrak{A}$. The actual transformation formulae, we will not explicitly state here, but they can easily be obtained from (II) and (III). We have the following corollary to Theorem 1:

COROLLARY. $\{x_{mi}\}_{m \text{ fixed}}$ *is a* D-*basis for* $\mathfrak{A}$.

$\{x_{mi}^-\}_{m \text{ fixed}}$ *is a* D^--*basis for* $\mathfrak{A}$.

Proof. A D-basis (see [2]) is a basis such that $Dx_{mi} = x_{mi-1}$. $Dx_{mi} = D_m K^m x_i = K^m Dx_i = K^m x_{i-1} = x_{mi-1}$ and $D^- x_{mi}^- = JDJJx_i = JDx_i = Jx_{i-1} = x_{mi-1}^-$ which is the condition for a D^--basis.

(5) $\mathrm{L} = \mathrm{S}$. In this case we have the following properties:

(*a*) $\mathrm{S}_m(1) = \mathrm{K}^m \mathrm{S} \mathrm{K}^{-m}(1) = m + x_1$ and $\mathrm{S}_m^-(1) = \mathrm{J}\mathrm{S}_m \mathrm{J}(1) = m - x_1$. It follows that $\mathrm{S}_m \neq \mathrm{S}_n$, $\mathrm{S}_m^- \neq \mathrm{S}_n^-$ if $m \neq n$ and $\mathrm{S}_m \neq \mathrm{S}_m^-$.

(*b*) $\mathrm{DS}_m = \mathrm{D}_m \mathrm{S}_m = (\mathrm{DS})_m = \mathrm{I}$ since $\mathrm{DS} = \mathrm{I}$ and $\mathrm{S}_m \mathrm{D} = \mathrm{S}_m \mathrm{D}_m = (\mathrm{SD})_m = (\sigma')_m = \sigma'_m$ since $\mathrm{SD} = \sigma'$, $\mathrm{D}^- \mathrm{S}_m^- = (\mathrm{DS}_m)^- = \mathrm{I}$ and $\mathrm{S}_m^- \mathrm{D}^- = (\sigma'_m)^- = \sigma'_{-m}$.

(*c*) Let b be any element of $\mathfrak{A}$ such that $\mathrm{D}b = a$ then $\mathrm{S}_m(a) = \mathrm{S}_m \mathrm{D} b = \sigma'_m(b) = b - \sigma_m(b)$ and similarly if b^- is such that $\mathrm{D}^- b^- = a$ then $\mathrm{S}_m^-(a) = b^- - \sigma_m^-(b)$.

(*d*) From (*c*) follows at once that $\mathrm{S}_m(a)$ and $\mathrm{S}_n(a)$ differ at most by an integer for any m and n. This integer however does depend on the choice of a. A similar statement holds for $\mathrm{S}_m^-(a)$ and $\mathrm{S}_n^-(a)$.

(*e*) Let R be any semi-integration on $\mathfrak{A}$, that is, a linear operator such that there exists a semi-derivation T and an algebra homorphism $\rho : \mathfrak{A} \to \mathbf{Z}$ such that $\mathrm{R}(a) = b - \rho(b)$ for any b for which $\mathrm{T}b = a$. From our preceding investigations it follows at once that $\mathrm{T} = \mathrm{D}$ or D^- and $\rho = \sigma_m$ for some m. From our definition it follows at once that—if T equals to, say, D—we have: $\mathrm{RD} = \mathrm{I} - \sigma_m$. Multiplying from the right by S_m, we get $\mathrm{R} = \mathrm{S}_m - \sigma_m \mathrm{S}_m = \mathrm{S}_m - (\sigma \mathrm{S})_m = \mathrm{S}_m$ since $\sigma \mathrm{S} = 0$. We collect this information in the following theorem:

THEOREM 2. *The only possible semi-integrations on* $\mathfrak{A}$ *are* S_m *and* S_m^- *which are all different from each other and conjugates of* S. *They have the following properties*:

$$\mathrm{DS}_m = \mathrm{I} \quad , \quad \mathrm{S}_m \mathrm{D} = \sigma'_m \quad \textit{and} \quad \mathrm{D}^- \mathrm{S}_m^- = \mathrm{I} \quad , \quad \mathrm{S}_m^- \mathrm{D}^- = \sigma'_{-m}$$

$\mathrm{S}_m(a)$ *and* $\mathrm{S}_n(a)$ *differ at most by an integer which does not depend on the choice of* m *and* n, *but does depend on the argument* a. *The same statement holds for* $\mathrm{S}_m^-(a)$ *and* $\mathrm{S}_n^-(a)$.

DEFINITION. $\{z_{mi}\}$ is called *a* S_m–D–basis of $\mathfrak{A}$, if it is a D–basis and if for each i we have $\mathrm{S}_m z_{mi} = z_{mi+1}$. A S_m^-–D^-–basis is defined analogeously.

COROLLARY. *The basis* $\{x_{mi}\}$ *are the only possible* S_m–D–*basis. The basis* $\{x_{mi}^-\}$ *are the only possible* S_m^-–D^-–*basis.*

Proof. $\mathrm{S}_m x_{mi} = \mathrm{K}^m \mathrm{S} x_i = \mathrm{K}^m x_{i+1} = x_{mi+1}$ and if z_{mi} is any S^m–D–basis then $z_{mi} = \mathrm{S}_m^{(i)}(1) = x_{mi}$ and a similar proof for the second statement in the corollary.

The following lemma is crucial for our further development of the theory. It shows that there is a natural pairing between the operators $\{\mathrm{S}_m\}$ and the operators $\{\mathrm{S}_m^-\}$ which will ultimately allow us to find the inverse operator of S_m on a suitable extension ring of $\mathfrak{A}$ and this inverse will satisfy the product rule of a semi-derivation.

DEFINITION. Let $\overline{S}_m = S^{-}_{1-m}$ and $D = D^{-}$.

LEMMA. $\overline{S}_m = S'_m$.

Proof. If $m = 0$ then $\overline{S}(x_i) = S^{-}_{1}(x_i) = ({}^1J)\, S\, ({}^1J)(x_i) = x_i - x_{i+1}$ by formula (III). On the other hand:

$$S'(x_i) = x_i - x_{i+1} \qquad \text{so that} \quad \overline{S} = S'.$$

Now $\overline{S}_m = S^{-}_{1-m} = [(S_1)_{-m}]^{-} = (S^{-}_{1})_m = (\overline{S})_m = (S')_m = S'_m$.

Given the basis $\{x_{mi}\}$ we associate with it the basis $\{\overline{x}_{mi}\}$ where $\overline{x}_{mi} = x^{-}_{1-mi}$ and call it its dual basis. Note that:

$$\overline{\overline{x}}_{mi} = x_{mi} \qquad \text{and} \quad \overline{S}_m\, \overline{x}_{mi} = \overline{x}_{mi+1} \qquad \text{and} \quad \overline{D}\,\overline{x}_{mi} = \overline{x}_{mi-1}.$$

LITERATURE

[1] E. G. KUNDERT (1966) – *Structure Theory in s–d–Rings.* Nota I, «Acc. Naz. Lincei», ser. VIII, *41*.

[2] T. GIEBUTOWSKI (1971) – *Ph. D. Thesis*, Univ. of Mass.