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On boundary conditions and fixed points for α-nonexpansive multivalued mappings

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Analisi funzionale. — *On boundary conditions and fixed points for α-nonexpansive multivalued mappings* (*). Nota di ESPEDITO DE PASCALE e RENATO GUZZARDI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema di punto fisso per mappe multivoche α–non-espansive con condizioni sulla frontiera che generalizzano la ben nota condizione al contorno di Leray–Schauder.

1. INTRODUCTION

The main purpose of this paper is to prove that a α-nonexpansive upper-semicontinuous multivalued map $f: B \multimap X$, where X is a Banach space and $B = B(0, r) = \{x \in X : \|x\| \leq r\}$, has a fixed point if the following three conditions hold:

i) $f(x)$ is convex for every $x \in B$.

ii) if $\lambda x \in f(x)$ for some $x \in \partial B$, then there exists $\beta \leq 1$ such that $\beta x \in f(x)$ (condition G).

iii) $(I - f)(B)$ is closed.

We shall employ the following three main theorems.

THEOREM A (L. Vietoris [1]). *Let $f: X \to Y$ be a continuous map such that $f(X) = Y$ and $f^{-1}(y)$ is acyclic for every $y \in Y$. If X and Y are compact metric spaces then $f_*: H_*(X) \to H_*(Y)$ is an isomorphism.*

We remark that Theorem A can be formulated in a more general setting. However the statement we have adopted is sufficient for our purposes.

THEOREM B (J. Dugundji [2]). *Any convex subset of a locally convex metrizable linear space is an absolute retract.*

THEOREM C (S. Eilenberg and D. Montgomery [3]). *Let X be a compact, acyclic absolute neighborhood retract and $f: X \multimap X$ an uppersemicontinuous multivalued map. Assume that $f(x)$ is acyclic for every $x \in X$. Then f has a fixed point.*

As particular cases of our theorem we obtain several well known fixed point theorems for multivalued and singlevalued maps.

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2. NOTATIONS AND DEFINITIONS

2.1 *Multivalued maps.*

We recall that a multivalued map f of a set X into a set Y is a triple (G , X , Y), where G, the graph of f, is a subset of $X \times Y$ such that $f(x) = \{y \in Y : (x , y) \in G\}$ is nonempty for each $x \in X$. $f(X) = \cup \{f(x) : x \in X\}$ is the range of f, while X is its domain.

We shall use the symbol $f : X \multimap Y$ to indicate a multivalued map and $f : X \to Y$ for the single-valued maps. If $A \subset X$ and $B \subset Y$ then $f(A) = \cup \{f(x) : x \in A\}$ while $f^+(B) = \{x \in X : f(x) \subset B\}$ and $f^-(B) = \{x \in X : f(x) \cap B \neq \varphi\}$. If $f : X \to Y$ we have $f^+(B) = f^-(B) = f^{-1}(B)$.

Let X and Y be topological spaces and $f : X \multimap Y$. We say that f is uppersemicontinuous at $x_0 \in X$ if for any open set O containing $f(x_0)$ there exists a neighborhood U of x_0 such that $x \in U(x_0)$ implies $f(x) \subset O$. If f is uppersemicontinuous at each point $x \in X$ and $f(x)$ is compact for every $x \in X$ then f is said to be uppersemicontinuous on X.

The following conditions are equivalent to uppersemicontinuity on X:

a) For any open set $O \subset Y$, the set $f^+(O)$ is open;
b) For any closed set $C \subset Y$, $f^-(C)$ is closed.

We say that a multivalued map $f : X \multimap Y$ is proper if for any compact set K contained in Y, $f^-(K)$ is compact. It is easy to see that a proper uppersemicontinuous multivalued map is closed. A fixed point of a multivalued map $f : X \multimap X$ is a point $x \in X$ such that $x \in f(x)$.

2.2 *Kuratowski measure of noncompactness.*

Let X be a metric space. For any bounded set $A \subset X$ we define $\alpha(A)$ (C. Kuratowski [7]) as the infimum of all $r > 0$ such that A can be covered by a finite number of subsets with diameter less than r. Let us recall here some properties of this number, called " measure of noncompactness ".

a) $\alpha(A) = 0$ if and only if A is precompact. If X is a Banach space;

b) $\alpha(\overline{\text{co}}\, A) = \alpha(A)$ where $\overline{\text{co}}\, A$ indicates the closure of the convex hull of A;

c) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ where $A + B = \{x + y : x \in A$ and $y \in B\}$;

d) For every positive real number t, $\alpha(tA) = t\alpha(A)$ where $tA = \{tx : x \in A\}$.

Let $f : X \multimap Y$ be an uppersemicontinuous map. The map f is said to be α-Lipschitz with constant $K \geq 0$, if for any bounded set $A \subset X$

$$\alpha(f(A)) \leq K\alpha(A) .$$

If $K < 1$, then f is called α-contraction; if $K = 1$, then f is called α-nonexpansive. If for any bounded subset $A \subset X$ such that $\alpha(A) \neq 0$, we have $\alpha(f(A)) < \alpha(A)$, then f is called condensing. The class of condensing maps is wider than the class of α-contractions as shown by Furi-Vignoli [4].

The map f is said to be completely continuous if it sends bounded sets into precompact sets.

2.3. *Homology*, AR *and* ANR *spaces.*

Let $\mathscr{T}$ be the category of topological spaces, $\mathscr{F}$ be the category of graded vector spaces over a field F. By $H_k(X)$, where $X \in \mathscr{T}$, we denote the K-th Vietoris homology vector space associated to X and by $H_*(X)$ the graded vector space associated to X. Given a continuous map $f: X \to Y$ we denote by $f_*: H_*(X) \to H_*(Y)$ the induced homomorphism.

A nonempty topological space X is said to the be acyclic if $H_i(X) = 0$ for $i \neq 0$ and $H_0(X) \simeq F$.

A nonempty topological space is said to be an absolute retract if for each homeomorphism h mapping X onto a closed subset of a metric space Y, the set $h(X)$ is a retract of Y. Similarly, a space X is said to be an absolute neighborhood retract, provided that for each homeomorphism h mapping X onto a closed subset of a metric space Y, $h(X)$ is a neighborhood retract in Y.

2.4. *Further notations.*

In what follows, unless otherwise stated, X will stand for a Banach Space, $B(0, r) = \{x \in X : \|x\| \leq r\}$, $\mathring{B} = \{x \in X : \|x\| < r\}$, $\partial B = B \setminus \mathring{B}$ and $\Pi : X \to B(0, r)$ will be the radial retraction of X onto $B(0, r)$.

We shall say that a map $f: B(0, r) -\!\!\circ\, X$ satisfies condition "P" if "$\lambda x \in f(x)$ for some $x \in \partial B$ implies $\lambda \leq 1$" and we shall say that f satisfies the weaker boundary condition "G" if "$\lambda x \in f(x)$ for some $x \in \partial B$ implies that there exists $\beta \leq 1$ such that $\beta x \in f(x)$".

In the following I denotes the identity map.

3. Results

LEMMA 3.1 (Martelli [8]). *Let $f: B(0, r) -\!\!\circ\, X$ be a condensing map with convex values.*

Then $\Pi \circ f(x)$ is acyclic for every $x \in B(0, r)$.

Proof. Since $f(x)$ is compact and Π is continuous, $\Pi \circ f(x)$ is compact. It is easy to see that $\Pi^{-1}(Y)$ is acyclic for every $y \in \Pi \circ f(x)$. Applying theorem A we obtain that

$$\Pi_* : H_*(f(x)) \to H_*(\Pi \circ f(x))$$

is an isomorphism. Since $f(x)$ is convex, $\Pi \circ f(x)$ is acyclic.

LEMMA 3.2 (Kuratowski [7]). *Let X a complete metric space and let $A_1 \supset A_2 \supset \cdots$ be a decreasing sequence of nonempty closed subsets of X. Assume that $\alpha(An)$ converges to 0. Then $A_\infty = \bigcap_{n \in N} A_n$ is nonempty and compact.*

LEMMA 3.3. *The radial retraction is α-nonexpansive.*

Proof. Let A be a bounded subset of X. Then $\Pi(A) \subset \overline{co}(A \cup \{o\})$. Since $\alpha(\overline{co}(A \cup \{o\})) = \alpha(A)$, it follows that $\alpha(\Pi(A)) \leq \alpha(A)$.

THEOREM 3.1. *Let $f: B \multimap E$ be an α-contraction with convex values. Let us assume that f satisfies the boundary condition "G". Then f has a fixed point.*

Proof. Since Π is α-nonexpansive, $\Pi \circ f : B \multimap B$ is an α-contraction. Moreover by Lemma 3.1, $\Pi(f(x))$ is acyclic for every $x \in B$.

We define inductively a sequence of sets: $B_0 = B$, $B_{n+1} = \overline{co}\, \Pi \circ f(B_n)$ for every $n \in N$. It is easily seen that $B_n \supset B_{n+1}$ for every $n \in N$ and $\alpha(B_n) \to 0$ as $n \to \infty$.

Let us put $B_\infty = \bigcap_{n \in N} B_n$. Then, because of Lemma 3.2, one has that B_∞ is nonempty and compact. Since $\Pi \circ f(B_\infty) \subset B_\infty$, because of theorem B and theorem C, $\Pi \circ f$ has a fixed point in B_∞. Let $x \in B$ such that $x \in \Pi(f(x)) = (f(x) \cap \mathring{B}) \cup \Pi(f(x) \setminus \mathring{B})$.

If $\|x\| < r$ then $x \in f(x) \cap \mathring{B}$. Thus $x \in f(x)$ and the statement is proved. Suppose $\|x\| = r$. We have $x \in \Pi(f(x)) \setminus \mathring{B})$. It follows that there exists $\lambda \geq 1$ such that $\lambda x \in f(x)$. Because of condition "G" there exists $\beta \leq 1$ such that $\beta x \in f(x)$. By the convexity of $f(x)$, we have that the segment joining λx with βx is entirely contained in $f(x)$, thus $x \in f(x)$.

Now let us turn to the main result.

THEOREM 3.2. *Let $f: B \multimap E$ be an α-nonexpansive map with convex values, such that $(I - f)(B)$ is closed. Let us assume that f satisfies condition G. Then f has fixed point.*

Proof. For each $n \in N$ let us consider the map $f_n : B \multimap E$ defined by $f_n(x) = \frac{n}{n+1} f(x)$. We have:

$$\alpha(f_n(A)) = \alpha\left(\frac{n}{n+1} f(A)\right) = \frac{n}{n+1}\, \alpha(f(A)) \leq \frac{n}{n+1}\, \alpha(A) < \alpha(A).$$

This implies that f_n is a α-contraction with constant $\frac{n}{n+1}$. Furthermore f_n satisfies condition G. In fact

$$\lambda x \in f_n(x) \quad \text{and} \quad x \in \partial B \Rightarrow \frac{(n+1)\lambda}{n} x \in f(x) \quad \text{e} \quad x \in \partial B \Rightarrow \exists \beta \leq 1 :$$
$$: \beta x \in f(x) \Rightarrow \frac{\beta n}{n+1} x \in f_n(x) \quad \text{with} \quad \frac{\beta n}{n+1} < 1 .$$

By Theorem 3.1 there exists an element $x_n \in B$ such that $x_n \in f_n(x_n)$. Consequently $x_n - \frac{n}{n+1} x_n \in (I - f)(B)$ and since x_n s bounded and $\frac{n}{n+1} \to 1$, as $n \to \infty$ we have $0 \in (I - f)(B)$ and the statement follows.

Remark 3.1. A careful inspection and suitable modifications of our proofs of Theorems 3.1 and 3.2 show that they hold also when f is acyclic-valued and the condition "G" is replaced by the condition "$\lambda x \in f(x)$ and $x \in \partial B$ implies $f(x)$ convex and there exists $\beta \leq 1$ such that $\beta x \in f(x)$".

Remark 3.2. Theorems 3.1 and 3.2 fail to be valid if one replaces the assumption "$f(x)$ convex for every $x \in B$" with the assumption "$f(x)$ acyclic for every $x \in B$". This is easily seen from the following simple counterexample, where $f(x)$ is contractible for every $x \in B$. Let us consider $B(0, r)$ in R^2-plane and the constant multivalued map $f: B \multimap R^2$ defined by $f(x) = \Gamma$, where Γ is the locus of points of R^2 satysfiyng the equations

$$\rho = r\vartheta \qquad \frac{\Pi}{2} \leq \vartheta \leq 2\Pi + \frac{\Pi}{2}$$

in the standard polar coordinates. Clearly the map f is a completely continuous map, that satisfies condition "G" and does not have fixed points. But the map f fails to be convex valued.

Remark 3.3. A subset A of a Banach space X is said to be star-shaped if there exists $y \in A$ such that the line segment joning y with every point of A is entirely contained in A.

We may leave open the following question: do Theorem 3.1 and 3.2 hold if the assumption "$f(x)$ convex for every $x \in B$" is replaced by "$f(x)$ star-shaped for every $x \in B$"?

It is known that if f is a single-valued condensing map then $I - f$ is closed. We generalize and extend this result to the context of condensing multivalued maps.

PROPOSITION 3.1. *Let* E *be a closed bounded subset of a Banach space* X *and* $f: E \multimap X$ *be a condensing map. Then* $I - f$ *is proper.*

Proof. Let $K \subset E$ be compact and set $A = (I - f)^{-}(K)$. Since $I - f$ is upper semicontinuous, A is closed in E. We also have $A \subset K + f(A)$.

Let us suppose that $\alpha(A) > 0$.

Then $\alpha(A) \leq \alpha(K) + \alpha(f(A)) < \alpha(A)$ which is impossible.

Hence $\alpha(A) = 0$ and A is compact.

COROLLARY 3.1 (Martelli [8]). *Let* $f: B \multimap X$ *be a condensing map with convex values. Let us assume that* f *satisfies condition* P. *Then* f *has a fixed point.*

Proof. Follows from Proposition 3.1.

COROLLARY 3.2 (A. Granas [5]). *Let* $f: B(0, r) \multimap X$ *be an uppersemicontinuous map with closed and convex values. Let us assume that* f *is compact and* $f(x) \subset B(0, r)$ *for every* $x \in \partial B$. *Then* f *has a fixed point.*

Theorem 2 contains also, as a particular case, the well known result of Röthe [10]. It contains also several other theorems which would be too long to mention here. As examples we will give only the following two.

COROLLARY 3.3 (M. Krasnoselskij [6]). *Let* $f : B(0, r) \to H$ *be a continuous compact map, where* H *is a Hilbert space. If for every* $x \in \partial B$

$$(f(x), x) \leq \| x \|^2,$$

then f *has a fixed point.*

COROLLARY 3.4 (W. V. Petryshyn [9]). *Let* $f : B(0, r) \to X$ *be a condensing map which satisfies condition* P. *Then* f *has a fixed point.*

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