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A Picone Identity for Elliptic Differential Operators of Order 4 m with Applications

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Equazioni differenziali. — *A Picone Identity for Elliptic Differential Operators of Order* $4m$ *with Applications.* Nota di NORIO YOSHIDA, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Nel presente articolo si stabilisce una identità del tipo di Picone per una classe di operatori ellittici a derivate parziali di ordine $4m$. La predetta identità è poi applicata per dimostrare teoremi di confronto del tipo di Sturm, e anche per ottenere diseguaglianze del tipo di Wirtinger e limitazioni inferiori per gli autovalori relativi agli operatori considerati con condizioni ai limiti omogenee.

1. INTRODUCTION

Suppose that u and v are, respectively, functions in the domains of the ordinary differential operators

$$lu \equiv (au')' + cu \quad , \quad Lv \equiv (Av')' + Cv ,$$

defined in the interval (x_1, x_2). If $v \neq 0$ in (x_1, x_2), then the following formula, known as Picone's identity, holds:

$$(1.1) \qquad \left[\frac{u}{v}(au'v - Auv')\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} [(a - A)\,u'^2 + (C - c)\,u^2]\,dx + \int_{x_1}^{x_2} A\left(u' - \frac{u}{v}v'\right)^2 dx + \int_{x_1}^{x_2} \frac{u}{v}(vlu - uLv)\,dx .$$

The principal application of (1.1) is in the proof of the Sturm comparison theorem which asserts that if $a \geqq A > 0, C \geqq c$, then every solution v of the equation $Lv = 0$ oscillates more rapidly than any solution u of the equation $lu = 0$.

Beginning with the work of Picone [12], extensions of (1.1) to elliptic partial differential operators have been obtained by various authors. We refer, in particular, to Dunninger [5], Kreith [7-9], Kreith and Travis [10] and Swanson [13] for extensions to second order elliptic operators, and to Chan and Young [2], Dunninger [6], Kusano and Yoshida [11], Wong [14] and Yoshida [15] for extensions to fourth order elliptic operators.

The object of this paper is to present a generalization of (1.1) to the elliptic differential operators

$$(1.2) \qquad \Delta^m(a\Delta^m u) + cu \quad , \quad \Delta^m(A\Delta^m v) + Cv ,$$

(*) Nella seduta dell'8 marzo 1975.

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplacian, and Δ^m is the m-th iterated Laplacian.

As an application of the generalized Picone identity we prove Sturmian comparison theorems for elliptic differential inequalities involving (1.2) under hypotheses different from those given by Diaz and Dunninger [4]. As further applications we obtain a Wirtinger-type inequality for solutions of the associated differential equations and lower bounds for the first eigenvalue of related eigenvalue problems. Our results constitute generalizations of those of Dunninger [6] for the special case of fourth order operators ($m = 1$).

2. A PICONE IDENTITY

Let Ω be a bounded domain in the real Euclidean n-space E^n, with a piecewise smooth boundary $\partial\Omega$. Points in E^n are denoted by $x = (x_1, \cdots, x_n)$ and partial differentiation with respect to x_i by D_i, $i = 1, \cdots, n$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with nonnegative integer components and norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, we define the differential operator D^α by $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. We use ∇u to denote the gradient of a function $u \in C^1(\overline{\Omega})$.

Consider the linear elliptic differential operators

$$lu \equiv \Delta^m (a \Delta^m u) + cu \quad , \quad Lv \equiv \Delta^m (A \Delta^m v) + Cv ,$$

where the real-valued functions a and A are positive in Ω and of class $C^{2m}(\overline{\Omega})$, and the real-valued functions c and C are continuous in $\overline{\Omega}$. The domains D_l and D_L of l and L, respectively, are defined to be the set of all real-valued functions of class $C^{4m}(\Omega) \cap C^{2m}(\overline{\Omega})$.

We present a generalization of Picone's identity (1.1) in the following sense.

THEOREM 2.1. *Assume $\partial\Omega$ is piecewise smooth. If $u \in D_l$, $v \in D_L$ and if $1/v$, $1/\Delta v, \cdots, 1/\Delta^{m-1} v$ are of class $C^1(\overline{\Omega})$, then*

$$(2.1) \quad \int_{\partial\Omega} \sum_{k=0}^{m-1} \frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v} \left[\Delta^{m-k-1} u \frac{\partial}{\partial \nu} (\Delta^k (A \Delta^m v)) - \Delta^{m-k-1} v \frac{\partial}{\partial \nu} (\Delta^k (a \Delta^m u)) \right] ds$$

$$+ \int_{\partial\Omega} \sum_{k=0}^{m-1} \left[\Delta^{m-k-1} (a \Delta^m u) \frac{\partial}{\partial \nu} (\Delta^k u) - \Delta^k (A \Delta^m v) \frac{\partial}{\partial \nu} \left(\frac{(\Delta^{m-k-1} u)^2}{\Delta^{m-k-1} v} \right) \right] ds$$

$$= \int_\Omega [(a - A)(\Delta^m u)^2 + (c - C) u^2] \, dx$$

$$+ \int_\Omega \left[A \left(\Delta^m u - \frac{\Delta^m v}{\Delta^{m-1} v} \Delta^{m-1} u \right)^2 + \right.$$

$$+\sum_{k=0}^{m-2} \frac{\Delta^{k+1}(A\Delta^m v)}{\Delta^{m-k-1} v}\left(\Delta^{m-k-1} u - \frac{\Delta^{m-k-1} v}{\Delta^{m-k-2} v}\,\Delta^{m-k-2} u\right)^2\Bigg]\,dx$$

$$-\,2\int_\Omega \sum_{k=0}^{m-1} \Delta^k(A\Delta^m v)\,\Delta^{m-k-1} v \left|\nabla\left(\frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v}\right)\right|^2 dx$$

$$+\int_\Omega \frac{u}{v}\,(u\mathrm{L}v - vlu)\,dx\,,$$

where $\partial/\partial\nu$ *denotes the exterior normal derivative.*

Proof. Let $\varphi_1,\cdots,\varphi_{2m}$ be real-valued functions such that $\varphi_1,\cdots,\varphi_m$ are of class $C^2(\overline{\Omega})$, $A\varphi_{m+1},\cdots,A\varphi_{2m}$ are of class $C^2(\overline{\Omega})$ and $\varphi_1,\cdots,\varphi_{m-1}$ do not vanish in $\overline{\Omega}$. Let $\psi_1,\cdots,\psi_m$ be real-valued functions of class $C^2(\overline{\Omega})$. Then, we have

$$(2.2)\qquad \int_\Omega \sum_{k=0}^{m-1}\sum_{i=1}^{n} D_i\,[D_i(A\varphi_{m+k})\,(\Delta^{m-k-1} u)^2 - D_i((\Delta^{m-k-1} u)^2)\,A\varphi_{m+k}]\,dx$$

$$=\int_\Omega \sum_{k=0}^{m-1} [(\Delta^{m-k-1} u)^2\,\Delta(A\varphi_{m+k}) - 2\,A\varphi_{m+k}(\Delta^{m-k-1} u)\,\Delta^{m-k} u\,-$$

$$-\,2\,A\varphi_{m+k}\,|\,\nabla(\nabla^{m-k-1} u)\,|^2]\,dx\,,$$

$$(2.3)\qquad 2\int_\Omega \sum_{k=0}^{m-1}\sum_{i=1}^{n} D_i\,[A\varphi_{m+k}(\Delta^{m-k-1} u)^2\,D_i\,\psi_{m-k}]\,dx$$

$$=2\int_\Omega \sum_{k=0}^{m-1} [(\Delta^{m-k-1} u)^2\,\nabla(A\varphi_{m+k})\cdot\nabla\psi_{m-k} + A\varphi_{m-k}(\Delta^{m-k-1} u)^2\,\Delta\psi_{m-k}\,+$$

$$+\,2\,A\varphi_{m+k}\,(\Delta^{m-k-1} u)\,\nabla\psi_{m-k}\cdot\nabla(\Delta^{m-k-1} u)]\,dx\,.$$

(Actually, the identities hold between the integrands). Using (2.2) and (2.3) we see that the following identity holds:

$$(2.4)\qquad \int_\Omega [A(\Delta^m u)^2 + Cu^2]\,dx$$

$$=\int_\Omega \Bigg[A(\Delta^m u - \varphi_m\,\Delta^{m-1} u)^2 + \sum_{k=0}^{m-2} \frac{A\varphi_{m+k+1}}{\varphi_{m-k-1}}\,(\Delta^{m-k-1} u\,-$$

$$-\,\varphi_{m-k-1}\,\Delta^{m-k-2} u)^2 + (A\varphi_{2m} + C)\,u^2\Bigg]\,dx$$

$$+\int_\Omega \sum_{k=0}^{m-1}\Bigg[\Delta(A\varphi_{m+k}) + A\varphi_{m+k}\,\varphi_{m-k} + 2\,\nabla(A\varphi_{m+k})\cdot\nabla\psi_{m-k}\,-$$

$$-\,\frac{A\varphi_{m+k+1}}{\varphi_{m-k-1}}\Bigg]\,(\Delta^{m-k-1} u)^2\,dx$$

$$- 2 \int_\Omega \sum_{k=0}^{m-1} A\varphi_{m+k} [| \nabla (\Delta^{m-k-1} u) |^2 - 2 (\Delta^{m-k-1} u) \nabla\psi_{m-k} \cdot \nabla (\Delta^{m-k-1} u) +$$

$$+ (- \Delta\psi_{m-k} + \varphi_{m-k}) (\Delta^{m-k-1} u)^2] \, dx$$

$$- \int_\Omega \sum_{k=0}^{m-1} \sum_{i=1}^{n} D_i [D_i(A\varphi_{m+k}) (\Delta^{m-k-1} u)^2 - D_i((\Delta^{m-k-1} u)^2) A\varphi_{m+k}] \, dx$$

$$- 2 \int_\Omega \sum_{k=0}^{m-1} \sum_{i=1}^{n} D_i [A\varphi_{m+k} (\Delta^{m-k-1} u)^2 D_i \psi_{m-k}] \, dx ,$$

where we have set $\varphi_0 = 1$. We note that the integrand of the third integral on the right hand side of (2.4) can be rewritten as follows.

$$\begin{aligned}
(2.5) \qquad & \sum_{k=0}^{m-1} A\varphi_{m+k} [| \nabla (\Delta^{m-k-1} u)|^2 - 2 (\Delta^{m-k-1} u) \nabla\psi_{m-k} \cdot \nabla (\Delta^{m-k-1} u) \\
& \quad + (- \Delta\psi_{m-k} + \varphi_{m-k}) (\Delta^{m-k-1} u)^2] \\
& = \sum_{k=0}^{m-1} A\varphi_{m+k} [| \nabla (\Delta^{m-k-1} u) - \Delta^{m-k-1} u \nabla\psi_{m-k} |^2 \\
& \quad + (- \Delta\psi_{m-k} - | \nabla\psi_{m-k} |^2 + \varphi_{m-k}) (\Delta^{m-k-1} u)^2] .
\end{aligned}$$

We now define the functions $\varphi_1 , \cdots, \varphi_{2m} , \psi_1 , \cdots, \psi_m$ by

$$\begin{aligned}
\varphi_k &= \Delta^k v/\Delta^{k-1} v , && 1 \leqq k \leqq m , \\
\varphi_{m+k} &= \Delta^k (A\Delta^m v)/(A\Delta^{m-k-1} v) , && 0 \leqq k \leqq m - 1 , \\
\varphi_{2m} &= \Delta^m (A\Delta^m v)/(Av) , && \\
\psi_k &= \log | \Delta^{k-1} v | , && 1 \leqq k \leqq m .
\end{aligned}$$

It is easy to verify that the following identities hold:

$$\begin{aligned}
(2.6) \qquad & \Delta (A\varphi_{m+k}) + A\varphi_{m+k} \varphi_{m-k} + 2 \nabla (A\varphi_{m+k}) \cdot \nabla\psi_{m-k} - \\
& \quad - A\varphi_{m+k+1}/\varphi_{m-k-1} = 0 , && 0 \leqq k \leqq m - 1 , \\
& - \Delta\psi_{m-k} - | \nabla\psi_{m-k} |^2 + \varphi_{m-k} = 0 , && 0 \leqq k \leqq m - 1 , \\
& A\varphi_{2m} + C = Lv/v , \\
& \nabla (\Delta^{m-k-1} u) - \Delta^{m-k-1} u \nabla\psi_{m-k} = \Delta^{m-k-1} v \nabla (\Delta^{m-k-1} u/\Delta^{m-k-1} v).
\end{aligned}$$

Substituting (2.6) into (2.4) and (2.5) and transforming the last two integrals with the use of Gauss' divergence formula, we obtain

$$(2.7) \qquad \int_\Omega [A (\Delta^m u)^2 + Cu^2] \, dx$$

$$= \int_\Omega \Bigg[A \left(\Delta^m u - \frac{\Delta^m v}{\Delta^{m-1} v} \Delta^{m-1} u\right)^2 +$$

$$+ \sum_{k=0}^{m-2} \frac{\Delta^{k+1}(A\Delta^m v)}{\Delta^{m-k-1} v} \left(\Delta^{m-k-1} u - \frac{\Delta^{m-k-1} v}{\Delta^{m-k-2} v} \Delta^{m-k-2} u\right)^2 + \frac{u^2}{v} Lv \Bigg] dx$$

$$- 2 \int_\Omega \sum_{k=0}^{m-1} \Delta^k (A\Delta^m v) \Delta^{m-k-1} v \left| \nabla \left(\frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v}\right)\right|^2 dx$$

$$+ \int_{\partial\Omega} \sum_{k=0}^{m-1} \Bigg[\Delta^k (A\Delta^m v) \frac{\partial}{\partial\nu} \left(\frac{(\Delta^{m-k-1} u)^2}{\Delta^{m-k-1} v}\right) -$$

$$- \frac{(\Delta^{m-k-1} u)^2}{\Delta^{m-k-1} v} \frac{\partial}{\partial\nu} (\Delta^k (A\Delta^m v))\Bigg] ds .$$

On the other hand, using an integral identity used in [3, p. 226] (or [4, p. 341]), we get

$$\int_\Omega ulu \, dx - \int_\Omega [a (\Delta^m u)^2 + cu^2] \, dx \tag{2.8}$$

$$= \int_{\partial\Omega} \sum_{k=0}^{m-1} \left[\Delta^{m-k-1} u \frac{\partial}{\partial\nu} (\Delta^k (a\Delta^m u)) - \Delta^{m-k-1} (a\Delta^m u) \frac{\partial}{\partial\nu} (\Delta^k u)\right] ds .$$

Now the desired Picone identity (2.1) follows readily by combining (2.7) with (2.8).

The following identity to which (2.1) reduces when $a = c = 0$ will often be useful.

THEOREM 2.2. *Assume* $\partial\Omega$ *is piecewise smooth. If* $u \in C^{2m}(\overline{\Omega})$, $v \in D_L$ *and if* $1/v$, $1/\Delta v, \cdots, 1/\Delta^{m-1} v$ *are of class* $C^1(\overline{\Omega})$, *then*

$$\int_{\partial\Omega} \sum_{k=0}^{m-1} \left[\Delta^k (A\Delta^m v) \frac{\partial}{\partial\nu} \left(\frac{(\Delta^{m-k-1} u)^2}{\Delta^{m-k-1} v}\right) - \frac{(\Delta^{m-k-1} u)^2}{\Delta^{m-k-1} v} \frac{\partial}{\partial\nu} (\Delta^k (A\Delta^m v))\right] ds \tag{2.9}$$

$$= \int_\Omega [A (\Delta^m u)^2 + Cu^2] \, dx - \int_\Omega \frac{u^2}{v} Lv \, dx$$

$$- \int_\Omega \Bigg[A \left(\Delta^m u - \frac{\Delta^m v}{\Delta^{m-1} v} \Delta^{m-1} u\right)^2 +$$

$$+ \sum_{k=0}^{m-2} \frac{\Delta^{k+1}(A\Delta^m v)}{\Delta^{m-k-1} v} \left(\Delta^{m-k-1} u - \frac{\Delta^{m-k-1} v}{\Delta^{m-k-2} v} \Delta^{m-k-2} u\right)^2\Bigg] dx$$

$$+ 2 \int_\Omega \sum_{k=0}^{m-1} \Delta^k (A\Delta^m v) \Delta^{m-k-1} v \left| \nabla \left(\frac{\Delta^{m-k-1} u}{\Delta^{m-k-1} v}\right)\right|^2 dx .$$

3. STURMIAN COMPARISON THEOREMS

We begin with the following comparison theorem.

THEOREM 3.1. *Assume that* $\partial\Omega$ *is piecewise smooth and that there exists a nontrivial function* $u \in D_l$ *which satisfies*

$$\int_\Omega ulu \, dx \leqq 0 , \tag{3.1}$$

$$D^\alpha u = 0 \quad \text{on } \partial\Omega , \quad |\alpha| \leqq 2m - 1 ,$$

$$V[u] \equiv \int_\Omega [(a - A)(\Delta^m u)^2 + (c - C) u^2] \, dx \geqq 0 .$$

If $v \in D_L$ *satisfies*

$$Lv \geqq 0 \quad \text{in } \Omega , \tag{3.2}$$

$$(-1)^k \Delta^k v > 0 \quad \text{at some point } x(k) \in \Omega , \quad 0 \leqq k \leqq m - 1 ,$$

$$(-1)^{m+k} \Delta^k (A \Delta^m v) \geqq 0 \quad \text{in } \Omega , \quad 0 \leqq k \leqq m - 2 ,$$

$$\Delta^{m-1} (A \Delta^m v) < 0 \quad \text{in } \Omega ,$$

then, at least one of the functions $v , \Delta v , \cdots , \Delta^{m-1} v$ *must vanish at some point of* $\overline{\Omega}$.

Proof. Suppose none of $v , \Delta v , \cdots , \Delta^{m-1} v$ vanish in $\overline{\Omega}$ Then, the Picone identity (2.1) is valid and readily implies, in view of (3.1) and (3.2) (observe that the second condition of (3.2) implies $(-1)^k \Delta^k v > 0$ throughout $\overline{\Omega}$, $0 \leqq k \leqq m - 1$), that

$$-2 \int_\Omega \Delta^{m-1} (A \Delta^m v) \, v \left| \nabla \left(\frac{u}{v} \right) \right|^2 dx \leqq 0 .$$

Consequently, $\nabla (u/v) = 0$ in Ω, that is, $u/v = \gamma$ in $\overline{\Omega}$ for some nonzero constant γ. However, this cannot happen since $u = 0$ on $\partial\Omega$ whereas $v > 0$ on $\partial\Omega$. Thus, at least one of $v , \Delta v , \cdots , \Delta^{m-1} v$ must vanish at some point of $\overline{\Omega}$.

We now proceed to obtain comparison theorems which are " strong " in the sense that the conclusions apply to Ω rather than $\overline{\Omega}$.

LEMMA 3.2. *Assume that* $\partial\Omega$ *is piecewise smooth and that there exists a nontrivial function* $u \in C^{2m}(\overline{\Omega})$ *which satisfies*

$$D^\alpha u = 0 \quad \text{on } \partial\Omega , \quad |\alpha| \leqq 2m - 2 , \tag{3.3}$$

$$M[u] \equiv \int_\Omega [A (\Delta^m u)^2 + C u^2] \, dx \leqq 0 .$$

Then, there does not exist a function $v \in D_L$ *which satisfies*

$$
\begin{aligned}
(3.4)\qquad & Lv \geqq 0 && \textit{in} \quad \Omega\,, \\
& v > 0 && \textit{on} \quad \partial\Omega\,, \\
& (-1)^k \Delta^k v > 0 && \textit{in} \quad \overline{\Omega}\,, \quad 1 \leqq k \leqq m-1\,, \\
& (-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 && \textit{in} \quad \Omega\,, \quad 0 \leqq k \leqq m-2\,, \\
& \Delta^{m-1} (A\Delta^m v) < 0 && \textit{in} \quad \Omega\,.
\end{aligned}
$$

Proof. Suppose there exists a function $v \in D_L$ which satisfies (3.4). Since $\Delta v < 0$ in Ω and $v > 0$ on $\partial\Omega$, the maximum principle implies that $v > 0$ in $\overline{\Omega}$. Hence, the identity (2.9) holds. Consequently, in view of (3.3) and (3.4), it follows from (2.9) that

$$
0 \geqq M[u] \geqq -2 \int_\Omega \Delta^{m-1} (A\Delta^m v)\, v \left| \nabla \left(\frac{u}{v}\right) \right|^2 dx \geqq 0\,.
$$

This implies that $u/v = \gamma$ in $\overline{\Omega}$ for some nonzero constant γ. However, this contradicts the fact that $u = 0$ on $\partial\Omega$ and $v > 0$ on $\partial\Omega$.

THEOREM 3.3. *Assume that* $\partial\Omega \in C^{2m}$ *and that there exists a nontrivial function* $u \in C^{2m}(\overline{\Omega})$ *such that*

$$
(3.5)\qquad D^\alpha u = 0 \quad \textit{on} \quad \partial\Omega\,, \qquad |\alpha| \leqq 2m-1\,,
$$

$$
(3.6)\qquad M[u] \leqq 0\,.
$$

Then, every $v \in D_L$ *which satisfies*

$$
\begin{aligned}
(3.7)\qquad & Lv \geqq 0 && \textit{in} \quad \Omega\,, \\
& v > 0 \quad \textit{at some point} && x(0) \in \Omega\,, \\
& (-1)^k \Delta^k v > 0 && \textit{in} \quad \overline{\Omega}\,, \quad 1 \leqq k \leqq m-1\,, \\
& (-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 && \textit{in} \quad \Omega\,, \quad 0 \leqq k \leqq m-2\,, \\
& \Delta^{m-1} (A\Delta^m v) < 0 && \textit{in} \quad \Omega
\end{aligned}
$$

must vanish at some point of Ω *unless* v *is a constant multiple of* u. *Moreover, if* $M[u] < 0$, *then* v *must vanish at some point of* Ω.

Proof. Our method is an adaptation of that used by Dunninger [6]. Without loss of generality we may suppose that $v \geqq 0$ on $\partial\Omega$. Then, by Lemma 3.2, $v = 0$ at some point of $\partial\Omega$. It follows from the maximum principle that $v > 0$ in Ω.

Since $\partial\Omega \in C^{2m}$ and u satisfies the boundary condition (3.5), u belongs to the Sobolev space $\overset{\circ}{H}_{2m}(\Omega)$ which is the closure in the norm

$$\| u \|_{2m} = \left[\int_{\Omega} \sum_{|\alpha| \leqq 2m} | D^{\alpha} u |^2 \, dx \right]^{1/2} \tag{3.8}$$

of the class $C_0^{\infty}(\Omega)$ of infinitely differentiable functions with compact support in Ω. (See e.g. Agmon [1, p. 131]). Let $\{u_\nu\}$ be a sequence of functions in $C_0^{\infty}(\Omega)$ converging to u in the norm (3.8). It is not difficult to see that the identity (2.9) holds for each pair $\{u_\nu , v\}$. Using (3.5), (3.6) and (3.7) in (2.9) we find $M[u_\nu] \geqq 0$. Since A and C are bounded, there is a positive constant K_1 such that

$$| M[u_\nu] - M[u] | \leqq K_1 \int_{\Omega} | \Delta^m u_\nu \Delta^m (u_\nu - u) + \Delta^m u \Delta^m (u_\nu - u) | \, dx +$$

$$+ K_1 \int_{\Omega} | u_\nu (u_\nu - u) + u (u_\nu - u) | \, dx ,$$

which yields with the use of the Schwarz inequality

$$| M[u_\nu] - M[u] | \leqq K_2 (\| u_\nu \|_{2m} + \| u \|_{2m}) \| u_\nu - u \|_{2m} , \tag{3.9}$$

where K_2 is a positive constant depending only on m and n. Since $\| u_\nu - u \|_{2m} \to 0$ as $\nu \to \infty$, it follows from (3.9) that $M[u_\nu] \to M[u]$. Since $M[u_\nu] \geqq 0$, we have $M[u] \geqq 0$, which, together with (3.6), implies that $M[u] = 0$.

Let B denote a ball with $\bar{B} \subset \Omega$ and define

$$Q_B[u_\nu] \equiv \int_B \left[\frac{\Delta^{m-1}(A \Delta^m v)}{\Delta v} \left(\Delta u_\nu - \frac{u_\nu}{v} \Delta v \right)^2 - 2 \Delta^{m-1} (A \Delta^m v) \, v \left| \nabla \left(\frac{u_\nu}{v} \right) \right|^2 \right] dx .$$

It is easy to verify that

$$Q_B[u_\nu] = \int_B \left[\frac{\Delta^{m-1}(A\Delta^m v)}{\Delta v} (\Delta u_\nu)^2 - \Delta^{m-1} (A \Delta^m v) \, \Delta \left(\frac{u_\nu^2}{v} \right) \right] dx .$$

Since $\Delta^{m-1} (A \Delta^m v)/\Delta v$ and $\Delta^{m-1} (A \Delta^m v)$ are bounded in B, there is a positive constant K_3 depending only on m, n and B such that

$$| Q_B[u_\nu] - Q_B[u] | \leqq K_3 \int_B | \Delta u_\nu \Delta (u_\nu - u) + \Delta u \Delta (u_\nu - u) | \, dx$$

$$+ K_3 \int_B | \Delta [u_\nu (w_\nu - w)] + \Delta [w (u_\nu - u)] | \, dx ,$$

where $w_\nu = u_\nu/v$ and $w = u/v$. Applying the Schwarz inequality we obtain from the above

$$| Q_B[u_\nu] - Q_B[u] | \leqq K_4(\| u_\nu \|_{2m,B} + \| u \|_{2m,B}) \| u_\nu - u \|_{2m,B} + \\ + K_4(\| u_\nu \|_{2m,B} \| w_\nu - w \|_{2m,B} + \| w \|_{2m,B} \| u_\nu - u \|_{2m,B}),$$

where K_4 is a positive constant depending only on m, n and B and the subscript B indicates the integrals involved in the norm (3.8) are to be taken over B only. Since $v > 0$ on $\overline{B}$, we see easily that $\| w_\nu - w \|_{2m,B} \to 0$ as $\| u_\nu - u \|_{2m,B} \to 0$, and hence $Q_B[u_\nu] \to Q_B[u]$ as $\nu \to \infty$. Since by (2.9)

$$0 \leqq Q_B[u_\nu] \leqq M[u_\nu]$$

and since $M[u_\nu] \to M[u] = 0$ as $\nu \to \infty$, it follows that $Q_B[u] = 0$, and consequently, $\nabla(u/v) = 0$ in B. Since B is arbitrary, we conclude that $u/v = \gamma$ in $\overline{\Omega}$ for some nonzero constant γ.

The proof of the second statement is immediate. This finishes the proof.

The following strong comparison theorem is our main result.

THEOREM 3.4. *Assume that* $\partial\Omega \in C^{2m}$ *and that there exists a nontrivial function* $u \in D_l$ *which satisfies*

$$\int_\Omega ulu \, dx \leqq 0, \tag{3.10}$$

$$D^\alpha u = 0 \quad on \quad \partial\Omega, \quad |\alpha| \leqq 2m - 1, \tag{3.11}$$

$$V[u] = \int_\Omega [(a - A)(\Delta^m u)^2 + (c - C) u^2] \, dx \geqq 0.$$

Then, every $v \in D_L$ *which satisfies*

$$\begin{aligned} &Lv \geqq 0 && in \ \Omega, \\ &v > 0 && at\ some\ point \ x(0) \in \Omega, \\ &(-1)^k \Delta^k v > 0 && in \ \overline{\Omega}, && 1 \leqq k \leqq m-1, \\ &(-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 && in \ \Omega, && 0 \leqq k \leqq m-2, \\ &\Delta^{m-1}(A\Delta^m v) < 0 && in \ \Omega, \end{aligned}$$

must vanish at some point of Ω *unless* v *is a constant multiple of* u. *Moreover, if* $V[u] > 0$, *then* v *must vanish at some point of* Ω.

Proof. The hypothesis $V[u] \geqq 0$ implies that

$$M[u] \leqq \int_\Omega [a(\Delta^m u)^2 + cu^2] \, dx. \tag{3.12}$$

From (2.8), (3.10) and (3.11) it follows that the right hand side of (3.12) is nonpositive. The conclusion follows from Theorem 3.3.

The proof of the following corollary is analogous to that of Dunninger [6], so we omit it.

COROLLARY 3.5. *Assume that* $\partial\Omega \in C^{2m}$, $a \geqq A \geqq 0$, $c \geqq C$ *in* Ω *and that* $u \in D_l$, $u \not\equiv 0$ *in any open subset of* Ω. *If either*

$$c \not\equiv C \quad \text{in} \quad \Omega$$

and u satisfies

$$\int_\Omega ulu \, dx \leqq 0,$$

$$D^\alpha u = 0 \quad \text{on} \quad \partial\Omega, \quad |\alpha| \leqq 2m - 1,$$

or

$$a > A \quad \text{and} \quad c \neq 0 \quad \text{at the same} \quad x_0 \in \Omega,$$

and u satisfies

$$lu = 0 \quad \text{in} \quad \Omega,$$

$$D^\alpha u = 0 \quad \text{on} \quad \partial\Omega, \quad |\alpha| \leqq 2m - 1,$$

then every $v \in D_L$ *which satisfies*

$$Lv \geqq 0 \quad \text{in} \quad \Omega,$$

$$v > 0 \quad \text{at some point} \quad x(0) \in \Omega,$$

$$(-1)^k \Delta^k v > 0 \quad \text{in} \quad \bar{\Omega}, \quad 1 \leqq k \leqq m-1,$$

$$(-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 \quad \text{in} \quad \Omega, \quad 0 \leqq k \leqq m-2,$$

$$\Delta^{m-1} (A\Delta^m v) < 0 \quad \text{in} \quad \Omega,$$

must vanish at some point of Ω.

4. WIRTINGER INEQUALITIES AND LOWER BOUNDS FOR EIGENVALUES

The following Wirtinger-type inequality which is an immediate consequence of Theorem 3.3, is a generalization of the earlier result of Dunninger [6]. For related results the reader is referred to Wong [14] and the references cited therein.

THEOREM 4.1. *Assume that* $\partial\Omega \in C^{2m}$ *and that there exists a function* $v \in D_L$ *which satisfies*

$$Lv = 0 \quad \text{in} \quad \Omega,$$

$$v > 0 \quad \text{in} \quad \Omega,$$

$$(-1)^k \Delta^k v > 0 \quad \text{in} \quad \bar{\Omega}, \quad 1 \leqq k \leqq m-1,$$

$$(-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 \quad \text{in} \quad \Omega, \quad 0 \leqq k \leqq m-2,$$

$$\Delta^{m-1} (A\Delta^m v) < 0 \quad \text{in} \quad \Omega.$$

Then, for every nontrivial function $u \in C^{2m}(\bar{\Omega})$ *which satisfies*

$$D^\alpha u = 0 \quad on \quad \partial\Omega\,, \qquad |\alpha| \leqq 2m - 1\,,$$

the following inequality holds:

$$\int_\Omega [A(\Delta^m u)^2 + Cu^2]\, dx \geqq 0\,,$$

where equality holds if and only if u is a constant multiple of v.

As a final application of the Picone identity we obtain a lower bound for the first eigenvalue of the eigenvalue problem

$$(4.1) \qquad lu = \lambda u \quad \text{in} \quad \Omega\,,$$
$$D^\alpha u = 0 \quad \text{on} \quad \partial\Omega\,, \qquad |\alpha| \leqq 2m - 1\,.$$

THEOREM 4.2. *Let* λ *be the first eigenvalue of the problem* (4.1) *and let* $u \in D_l$ *be an associated eigenfunction. Suppose there exists a function* $v \in D_L$ *which satisfies*

$$(-1)^k \Delta^k v > 0 \quad in \quad \bar{\Omega}\,, \qquad 0 \leqq k \leqq m - 1\,,$$
$$(-1)^{m+k} \Delta^k (A\Delta^m v) \geqq 0 \quad in \quad \Omega\,, \qquad 0 \leqq k \leqq m - 1\,.$$

If

$$V[u] = \int_\Omega [(a - A)(\Delta^m u)^2 + (c - C)\, u^2]\, dx \geqq 0\,,$$

then

$$\lambda \geqq \inf_{x \in \Omega} \left(\frac{Lv}{v}\right).$$

Proof. The conclusion readily follows, since (2.1) together with the above hypotheses implies that

$$\lambda \int_\Omega u^2\, dx - \int_\Omega u^2 \left(\frac{Lv}{v}\right) dx \geqq 0\,.$$

Remark. It is easy to observe that in Theorem 3.1 and Theorem 4.2 the boundary condition for u can be replaced by

$$\Delta^k u = \Delta^{m-k-1}(a\Delta^m u) + \alpha_k \frac{\partial}{\partial\nu}(\Delta^k u) = 0 \quad \text{on} \quad \partial\Omega\,, \qquad 0 \leqq k \leqq m - 1\,,$$

where $0 \leqq \alpha_k \leqq +\infty$ $\left(\alpha_k = +\infty\right.$ denotes the boundary condition $\left.\frac{\partial}{\partial\nu}(\Delta^k u) = 0\right)$.

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