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Totally Real Stibmanifolds of Complex Manifolds

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Geometria differenziale. — *Totally Real Submanifolds of Complex Manifolds.* Nota di GERALD D. LUDDEN (*), MASAFUMI OKUMURA e KENTARO YANO, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Si approfondisce lo studio di certe sottovarietà di una varietà complessa, com'é specificato nella seguente Introduzione.

§ 1. INTRODUCTION

There have been many papers studying complex submanifolds of complex manifolds, especially of complex space forms (see [6] for a survey of results and references). Recently there have been a number of papers concerning arbitrary submanifolds of complex manifolds (see [1], [2], [4], [5], [7], [9]). In particular, Chen and Ogiue [2] have studied submanifolds M of $\mathbf{M}$ such that $T_x(M) \cap \mathbf{J}T_x(M) = \{0\}$ for each x in M. The purpose of this paper is to study these submanifolds further. In particular in § 2 we consider the basic properties of such submanifolds and in § 3 we examine the Laplacian of the square of the length of the second fundamental form and prove a pinching theorem. § 4 is devoted to the study of parallel isoperimetric normal sections on these submanifolds.

§ 2. FUNDAMENTAL PROPERTIES

Let $\mathbf{M}$ be a Hermitian manifold of complex dimension m and let $\mathbf{J}$ be the almost complex structure and $\boldsymbol{g}$ the Hermitian metric on $\mathbf{M}$. Let M be an n-dimensional submanifold immersed in $\mathbf{M}$ satisfying $T_x(M) \cap \mathbf{J}T_x(M) = \{0\}$ for each $x \in M$, where $T_x(M)$ is the tangent space to M at x. Here we have identified $T_x(M)$ with its image under the differential of the immersion. We call such a submanifold M *totally real* or *anti-invariant*. If X is a vector field on M, we see $\mathbf{J}X$ is a vector field in the normal bundle of M. If ξ is a vector field in the normal bundle put

$$\mathbf{J}\xi = P\xi + Q\xi\,, \tag{1}$$

where $P\xi$ is the tangential part of $\mathbf{J}\xi$ and $Q\xi$ the normal part. Then P is a tangent bundle valued 1-form on the normal bundle and Q is an endomorphism of the normal bundle. Applying $\mathbf{J}$ to (1) and $\mathbf{J}X$ and comparing and normal parts, we have

$$PQ\xi = 0\,, \tag{2}$$

$$Q^2\xi = -\xi - \mathbf{J}P\xi\,, \tag{3}$$

$$P\mathbf{J}X = -X\,, \tag{4}$$

$$Q\mathbf{J}X = 0\,, \tag{5}$$

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(**) Nella seduta dell'8 marzo 1975.

where X is an arbitrary tangent vector field to M and ξ an arbitrary normal vector field. From (3) and (5) we see $Q^3 + Q = 0$ on the normal bundle (see [10]). We also see that $n \leq m$ since $\mathbf{J}$ is non-singular.

Let $\boldsymbol{\nabla}$ be the Riemannian connection of $\boldsymbol{g}$. Then, the Gauss and Weingarten equations are

$$(6) \qquad \boldsymbol{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad (7) \qquad \boldsymbol{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi .$$

Here ∇ is the Riemannian connection of the metric g induced on M from $\boldsymbol{g}$ (i.e. $g(X, Y) = \boldsymbol{g}(X, Y)$), σ is the second fundamental form of the immersion, $\nabla^\perp$ is the connection on the normal bundle induced from $\boldsymbol{\nabla}$ and $g(A_\xi X, Y) = \boldsymbol{g}(\sigma(X, Y), \xi)$. A vector field ξ in the normal bundle is *parallel* if $\nabla^\perp \xi = 0$. M is *totally geodesic* if $\sigma \equiv 0$. M is *minimal* if $\sum_{i=1}^{n} \sigma(e_i, e_i) = 0$ for any local ortho-normal basis $\{e_1, \cdots, e_n\}$ of tangent vectors to M.

Assume now that $\mathbf{M}$ is Kaehler (i.e. $\boldsymbol{\nabla}\mathbf{J} = 0$). Differentiating $\mathbf{J}$X and (1) and comparing tangential and normal parts we have

$$(8) \qquad -A_{\mathbf{J}Y} X = P\sigma(X, Y), \qquad (9) \qquad \nabla^\perp_X(\mathbf{J}Y) = \mathbf{J}\nabla_X Y + Q\sigma(X, Y),$$

$$(10) \qquad P\nabla^\perp_X \xi = \nabla_X(P\xi) - A_{Q\xi} X, \qquad (11) \qquad -\mathbf{J}A_\xi X + Q\nabla^\perp_X \xi = \sigma(X, P\xi) + \nabla^\perp_{X(Q\xi)} .$$

From (8) we have

$$-g(A_{\mathbf{J}Y} X, Z) = g(P\sigma(X, Y), Z),$$

or

$$-g(\sigma(X, Z), \mathbf{J}Y) = g(P\sigma(X, Y), Z).$$

If M is totally umbilical, that is $\sigma(X, Y) = g(X, Y) H$ for some normal vector field H, then $-g(X, Z)\,\boldsymbol{g}(H, \mathbf{J}Y) = g(X, Y)\,g(PH, Z)$. Letting $X = Z$ and $Y = PH$ we have $g(X, X)\,g(PH, PH) = g(X, PH)^2$. Now every real curve ($n = 1$) in $\mathbf{M}$ is totally real. If $n > 1$ we see that $PH = 0$. If $n = m$, then $Q = 0$ and $\mathbf{J} = P$. Thus we have

PROPOSITION 1. *If* $n = m > 1$ *and* M *is totally umbilical, then* M *is totally geodesic.*

Suppose now that $\mathbf{M}$ is a complex space form of constant holomorphic curvature $\boldsymbol{c}$. Denote $\mathbf{M}$ by $\mathbf{M}(\boldsymbol{c})$. Then the curvature operator $\mathbf{R}$ of $\mathbf{M}(\boldsymbol{c})$ assumes the form,

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \boldsymbol{c}/4\,\{\boldsymbol{g}(\mathbf{Y}, \mathbf{Z})\mathbf{X} - \boldsymbol{g}(\mathbf{X}, \mathbf{Z})\mathbf{Y} + (\mathbf{J}\mathbf{Y}, \mathbf{Z})\,\mathbf{J}\mathbf{X} - \boldsymbol{g}(\mathbf{J}\mathbf{X}, \mathbf{Z})\,\mathbf{J}\mathbf{Y} + 2\,\boldsymbol{g}(\mathbf{X}, \mathbf{J}\mathbf{Y})\,\mathbf{J}\mathbf{Z}\}.$$

If M is totally real then

$$(12) \qquad \mathbf{R}(X, Y)Z = \boldsymbol{c}/4\,\{g(Y, Z)X - g(X, Z)Y\}$$

which is tangent to M. On the other hand, if $\mathbf{R}(X, Y)Z$ is tangent to M for all X, Y, Z and $\boldsymbol{c} \neq 0$ then we obtain $\boldsymbol{g}(JX, Y)JX$ is tangent to M for all X, Y. Thus we have

PROPOSITION 2. ([2]). *If* $\mathbf{R}(X, Y)Z$ *is tangent to* M *for all* X, Y *and* $\boldsymbol{c} \neq 0$, *then* M *is a complex submanifold or is totally real.*

If M is totally real, the equations of Gauss and Codazzi become

$$g(R(X, Y)Z, W) = \boldsymbol{c}/4\,\{g(X, W)\,g(Y, Z) - g(X, Z)\,g(Y, W)\} + \\ + \boldsymbol{g}(\sigma(X, W), \sigma(Y, Z)) - \boldsymbol{g}(\sigma(X, Z), \sigma(Y, W)) \tag{13}$$

and

$$(\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z) = 0\,, \tag{14}$$

where

$$(\nabla_X \sigma)(Y, Z) = \nabla_X^{\perp}(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\,.$$

PROPOSITION 3 ([2]). *If* M *is a totally real, totally geodesic submanifold of a complex space form,* $\mathbf{M}(\boldsymbol{c})$, *then* M *is of constant curvature* $\boldsymbol{c}/4$.

COROLLARY 4. *If* $n = m > 1$ *and* M *is totally real and totally umbillical in a complex space form* $\mathbf{M}(\boldsymbol{c})$, *then* M *is of constant curvature* $\boldsymbol{c}/4$.

From equation (13) we see that

$$S(X, Y) = (n-1)\,\boldsymbol{c}/4\,g(X, Y) + \\ + \sum_i \{\boldsymbol{g}(\sigma(e_i, e_i), \sigma(X, Y) - \boldsymbol{g}(\sigma(e_i, X), \sigma(e_i, Y))\} \tag{15}$$

and

$$\rho = n(n-1)\,\boldsymbol{c}/4 + \\ + \sum_{i,j} \{\boldsymbol{g}(\sigma(e_i, e_i), \sigma(e_j, e_j)) - \boldsymbol{g}(\sigma(e_i, e_j), \sigma(e_i, e_j))\}\,, \tag{16}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal basis of tangent vectors to M. Here S is the Ricci tensor of M and ρ is the scalar curvature of M. If we let $\sigma(X, Y) = h^{\lambda}(X, Y)\,\xi_{\lambda}$, where $\{\xi_{\lambda}\}$ is a local ortho-normal basis of normal vectors to M, then (15) and (16) become

$$S(X, Y) = (n-1)\,\boldsymbol{c}/4\,g(X, Y) + \\ + \sum_{\lambda} \left\{ (tr\,h^{\lambda})\,h^{\lambda}(X, Y) - \sum_i h^{\lambda}(e_i, X)\,h^{\lambda}(e_i, Y) \right\}, \tag{15'}$$

and

$$\rho = n(n-1)\,\boldsymbol{c}/4 + \sum_{\lambda} (tr\,h^{\lambda})^2 - \|\sigma\|^2, \tag{16'}$$

respectively, where $tr\,h^{\lambda}$ is the trace of h^{λ}.

PROPOSITION 5 ([2]). *If* M *is a minimal totally real submanifold of a complex space form, then*

1) $S - (n-1)\,c/4\,g$ *is negative semi-definite,*

2) $\rho \leq n(n-1)\,c/4$.

M *is totally geodesic if and only if any of the following conditions are satisfied:*

1) $\rho = n(n-1)\,c/4$,

or

2) $S = (n-1)\,c/4\,g$,

or

3) $g(R(X,Y)Z, W) = c/4\,\{g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,W)\}$.

Ricci's equation is

$$(17) \qquad \mathbf{g}(\mathbf{R}(X,Y)\xi,\zeta) = \mathbf{g}(R^N(X,Y)\xi,\zeta) - g([A_\xi, A_\zeta]X, Y),$$

where $R^N(X,Y) = [\nabla^\perp_X \nabla^\perp_Y] - \nabla^\perp_{[X,Y]}$. Since $\mathbf{M}$ is a complex space form we see that

$$\mathbf{g}(\mathbf{R}(X,Y)\xi,\zeta) = c/4\,\{g(Y, P\xi)\,g(X, P\zeta) - g(X, P\xi)\,g(Y, P\zeta)\}.$$

Thus (17) becomes

$$(17') \qquad c/4\,\{g(Y, P\xi)\,g(X, P\zeta) - g(X, P\xi)\,g(Y, P\zeta)\} = \\ = \mathbf{g}(R^N(X,Y)\xi,\zeta) - g([A_\xi, A_\zeta]X, Y).$$

If $n = m$, then $Q = 0$ and $P = \mathbf{J}$. Also if ξ is a normal vector to M then $\xi = \mathbf{J}Z$ for some vector Z tangent to M. Thus, from (9) we see that

$$\nabla^\perp_X \xi = \nabla^\perp_X(\mathbf{J}Z) = \mathbf{J}\nabla_X Z.$$

This implies that $R^N(X,Y)\xi = \mathbf{J}R(X,Y)Z$. In this case (17') becomes

$$(18) \qquad g(R(X,Y)Z, W) = \\ = c/4\,\{g(Y, \mathbf{J}\xi)\,g(X, \mathbf{J}\zeta) - g(X, \mathbf{J}\xi)\,g(X, \mathbf{J}\zeta)\} + g([A_\xi, A_\zeta]X, Y),$$

where $\xi = \mathbf{J}Z$ and $\zeta = \mathbf{J}W$. Thus we have the following.

THEOREM 6. *Let* M *be a totally real submanifold of dimension* n *of a complex space form* $\mathbf{M}(\mathbf{C})$ *of a complex dimension* n. *If* $[A_\xi, A_\zeta] = 0$ *for any normal vectors* ξ *and* ζ *then* M *is of constant curvature* $c/4$. *If in addition,* M *is minimal then* M *is totally geodesic.*

Proof. The first statement follows from equation (18). For the second statement, comparing (18) and (13) we see

$$\boldsymbol{g}(\sigma(X, W), \sigma(Y, Z)) - \boldsymbol{g}(\sigma(X, Z), \sigma(Y, W)) = 0$$

for all tangent vectors X, Y, Z, W to M. Picking an orthonormal basis $\{e_j\}$ of the tangent vectors to M and letting $X = W = e_i$ and $Y = Z = e_j$ and summing over i we see $\sigma(e_i, e_j) = 0$ for all i and j. Thus the proof is done.

THEOREM 7. *If* M *is as in Theorem 6, then* $R^N \equiv 0$ *if and only if* $R \equiv 0$.

§ 3. LAPLACIAN OF $\| \sigma \|^2$

The purpose of this section is to prove the following.

THEOREM 8. *Let* M *be a compact totally real minimal submanifold of dimension* n *of a complex space form* $\mathbf{M}(\boldsymbol{c})$ *of complex dimension* m *and* $\boldsymbol{c} > 0$. *If*

$$\| \sigma \|^2 \leq \frac{n}{2 - \frac{1}{p}} \boldsymbol{c}/4 ,$$

where $p = 2m - n$, *then* M *is totally geodesic. A local theorem is obtained by replacing the condition that* M *is compact by* M *having constant scalar curvature.*

Proof. Let $\{e_1, \cdots, e_n\}$ be a local ortho-normal basis for the tangent vectors to M and $\{\xi_1 = \mathbf{J}e_1, \cdots, \xi_n = \mathbf{J}e_n, \xi_{n+1}, \cdots, \xi_p\}$ a local ortho-normal basis for the normal vectors to M. Then, from Proposition 3.5 of [2] we have

$$\frac{1}{2}\Delta \| \sigma \|^2 = \| \nabla\sigma \|^2 + \sum_{\lambda,\nu=1}^{p} tr(A_\lambda A_\nu - A_\nu A_\lambda)^2 - \sum_{\lambda,\nu=1}^{p} (tr A_\lambda A_\nu)^2 + n\boldsymbol{c}/4 \| \sigma \|^2 + \boldsymbol{c}/4 \sum_{\alpha=1}^{n} tr A_\alpha^2 , \tag{19}$$

where $A_\lambda = A_{\xi_\lambda}$ and Δ is the Laplacian operator.

We have the following lemma from [3].

LEMMA 9. *Let* A *and* B *be symmetric* $(n \times n)$*-matrices. Then*

$$- tr(AB - BA)^2 \leq 2\, tr A^2\, tr B^2 .$$

Applying Lemma 9 to (19) and proceeding as in [9] we have

$$\frac{1}{2}\Delta \| \sigma \|^2 \geq \| \nabla\sigma \|^2 - 2 \sum_{\lambda \neq \nu} tr A_\lambda^2\, tr A_\nu^2 - \Sigma (tr A_\lambda A_\nu)^2 + n\boldsymbol{c}/4 \| \sigma \|^2 + \boldsymbol{c}/4\, \Sigma\, tr A_\alpha^2 =$$

$$= \| \nabla\sigma \|^2 + c/4 \,\Sigma\, tr\, A_\alpha^2 + nc/4 \| \sigma \|^2 - 2 \sum_{\lambda<\nu} tr\, A_\lambda^2\, tr\, A_\nu^2 - (\Sigma\, tr\, A_\lambda^2)^2$$
$$= \| \nabla\sigma \|^2 + c/4 \,\Sigma\, tr\, A_\alpha^2 + nc/4 \| \sigma \|^2 - p^2 \sigma_1^2 - p(p-1)\sigma_2$$
$$= \| \nabla\sigma \|^2 + c/4 \,\Sigma\, tr\, A_\alpha^2 + nc/4 \| \sigma \|^2 - (2p^2 - p)\sigma_1^2 + p(p-1)(\sigma_1^2 - \sigma_2)$$
$$= \| \nabla\sigma \|^2 + c/4 \,\Sigma\, tr\, A_\alpha^2 + nc/4 \| \sigma \|^2 + p(p-1)(\sigma_1^2 - \sigma_2) - \left(2 - \frac{1}{p}\right) \| \sigma \|^4$$
$$\geq \left[nc/4 - \left(2 - \frac{1}{p}\right) \| \sigma \|^2\right] \| \sigma \|^2,$$

where $p\sigma_1 = \Sigma\, tr\, A_\lambda^2$ and $p(p-1)\sigma_2 = 2 \sum_{\lambda<\nu} tr\, A_\lambda^2\, tr\, A_\nu^2$. This holds since we can assume $tr\,(A_\lambda A_\nu) = 0$ if $\lambda \neq \nu$ and $p^2(p-1)(\sigma_1^2 - \sigma_2) = \sum_{\lambda<\nu} (tr\, A_\lambda^2 - tr\, A_\nu^2)^2 \geq 0$. If $nc/4 - \left(2 - \frac{1}{p}\right) \| \sigma \|^2 \geq 0$ then we see that $\Delta \| \sigma \|^2 \geq 0$. If M is compact, the well known lemma of E. Hopf says that $\Delta \| \sigma \|^2 = 0$. Also, note that if the scalar curvature ρ of M is constant then (16′) shows that $\| \sigma \|^2$ is constant and hence $\Delta \| \sigma \|^2 = 0$. From the above equations, wee that $\Delta \| \sigma \|^2 = 0$ implies that $\nabla\sigma = 0$, $\Sigma\, tr\, A_\alpha^2 = 0$ and $\sum_{\lambda<\nu} (tr\, A_\lambda^2 - tr\, A_\nu^2)^2 = 0$. Thus $A_\lambda = 0$ for all **c** and hence M is totally geodesic.

COROLLARY 10 ([2]). *Let* M *be a compact, minimal, totally real submanifold of dimension n of a complex space form* **M** (**c**), **c** > 0, *of complex dimension n. If*

$$\| \sigma \|^2 < \frac{n(n+1)}{(2n-1)}\, c/4$$

then M *is totally geodesic.*

Proof. In this case $\Sigma\, tr\, A_\alpha^2 = \| \sigma \|^2$ and $p = n$ so that the inequality in the above proof becomes

$$\tfrac{1}{2} \Delta \| \sigma \|^2 \geq \| \nabla\sigma \|^2 + (n+1)\, c/4 \| \sigma \|^2 + p(p-1)(\sigma_1^2 - \sigma_2) -$$
$$- \left(2 - \frac{1}{n}\right) \| \sigma \|^4 \geq \left[(n+1)\, c/4 - \left(2 - \frac{1}{n}\right) \| \sigma \|^2\right] \| \sigma \|^2 .$$

Again we see by Hopf's lemma $\Delta \| \sigma \|^2 = 0$ so $\| \sigma \| = 0$.

Remark. In Corollary 10 the condition is a strict inequality. The authors will consider equality in a forthcoming paper.

§ 4. PARALLEL ISOPERIMETRIC SECTIONS

A section ξ of the normal bundle is called *isoperimetric* if $tr\, A_\xi$ is constant.

Let M be a totally real submanifold of a complex space form **M** (**c**).

Now we can write equation (14) as

$$(14') \qquad \Sigma \{(\nabla_X h^\lambda)(Y, Z) - (\nabla_Y h^\lambda)(X, Z)\} \xi_\lambda + \\ + \Sigma \{h^\lambda (Y, Z) \nabla_X^\perp \xi_\lambda - h^\lambda (X, Z) \nabla_Y^\perp \xi_\lambda\} = 0,$$

or, if we let $\nabla_X^\perp \xi_\lambda = \Sigma L_{\lambda\nu}(X) \xi_\nu$, as

$$(14'') \qquad (\nabla_X A_\lambda) Y - (\nabla_Y A_\lambda) X - \Sigma \{L_{\lambda\nu}(X) A_\nu Y - L_{\lambda\nu}(Y) A_\nu X\} = 0.$$

If ξ is a parallel normal section then we can assume ξ is a unit vector field since its length is constant. Denote a unit parallel normal section by ξ_1 and use it as the first vector in an local ortho-normal basis of normal vectors. Then $L_{1\nu}$ are all zero and so (14″) gives $(\nabla_X A_1) Y = (\nabla_Y A_1) X$. From equation (17′) we see that

$$(20) \qquad [A_1, A_\lambda] X = c/4 \{g(X, P\xi_1) P\xi_\lambda - g(X, P\xi_\lambda) P\xi_1.$$

Let $f_1 = \| A_1 \|^2$. After a long calculation similar to that in [8], we find

$$(21) \qquad \tfrac{1}{2} \Delta f_1 = \| \nabla A_1 \|^2 + c/4 \{n \, tr \, A_1^2 - (tr \, A_1)^2\} + \\ + \Sigma \{tr \, A_\lambda \, tr (A_1^2 A_\lambda) - (tr \, A_1 A_\lambda)^2\}.$$

The following lemma appears in [8].

LEMMA 11. *Let* $A_1, \cdots, A_p$ *be a symmetric linear transformations of an n-dimensional inner product space* V. *Assume that* $[A_1, A_\lambda] = 0$ *for* $\lambda = 1, \cdots, p$. *If* $\{e_1, \cdots, e_n\}$ *is an orthonormal basis of* V *for which* $A_1 e_i = \lambda_i e_i$ *for* $i = 1, \cdots, n$ *then*

$$\Sigma \{tr \, A_\lambda \, tr (A_1^2 A_\lambda) - (tr \, A_1 A_\lambda)^2\} + nc \, tr \, A_1^2 - c (tr \, A_1)^2 = \\ = \sum_{i<j} \left\{ c + \sum_\lambda [a_{ii}^\lambda a_{jj}^\lambda - (a_{ij}^\lambda)^2] \right\} (\lambda_i - \lambda_j)^2,$$

where (a_{ij}^λ) *is the matrix of* A_λ.

We shall use these facts to prove the following.

THEOREM 12. *Let* M *be a compact totally real submanifold of a complex space form* $\mathbf{M}(c)$. *If* M *has non-negative sectional curvature and admits a parallel, isoperimetric normal section* ξ *such that* $P\xi = 0$ *and* $A\xi$ *has n distinct eigenvalues everywhere on* M, *then* M *is flat.*

Proof. From (20) we see that $P\xi_1 = 0$ implies $[A_1, A_\lambda] = 0$ for all λ. Thus we can apply Lemma 11 to (21) and obtain

$$\tfrac{1}{2} \Delta f_1 = \| \nabla A_1 \|^2 + \sum_{i<j} K_{ij} (\lambda_i - \lambda_j)^2,$$

where K_{ij} is the sectional curvature of the section spanned by $\{e_i, e_j\}$ and λ_i are the eigenvalues of A_1. Since the K_{ij} are non-negative we have that $\Delta f_1 \geq 0$ so that Hopf's lemma says $\Delta f_1 = 0$. Thus since $\lambda_i - \lambda_j \neq 0$ for $i \neq j$ we have $K_{ij} = 0$ and the proof is done.

COROLLARY 13. *Let* M *be a compact totally real surface immersed in a complex space form* $\mathbf{M}(c)$ *of complex dimension* > 2. *If the Gaussian curvature of* M *is non-negative and* M *admits a parallel, isoperimetric, umbillic-free normal section then* M *is flat.*

Remark. A generalization of Corollary 13 appears in [2].

THEOREM 14. *Let* M *be a compact, minimal, totally real submanifold of a complex space form* $\mathbf{M}(c)$. *Suppose*

1) *the real dimension* n *of* M *is less than the complex dimension* m *of* $\mathbf{M}$,
2) $R^N \equiv 0$ *on* $\mathbf{M}$.

Then there exist $2m - 2n$ *parallel isoperimetric, sections on* M *and if one of these sections has* n *distinct eigenvalues everywhere on* M *and the sectional curvature of* M *is non-negative then* M *is flat.*

Proof. This follows from known facts and Theorem 13.

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