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Alexander-Spanier cohomology of higher order

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Topologia. — *Alexander-Spanier cohomology of higher order.* Nota di NICOLAE TELEMAN (*), presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Per ogni numero naturale $k \geq 1$ definiamo un funtore coomologico $H^*_{(k)}(-, G)$ sulla categoria degli spazi topologici. Quando $k = 1$ si ottiene la coomologia di Alexander–Spanier.

Costruiamo una successione spettrale che converge verso la coomologia introdotta, successione spettrale che generalizza la successione spettrale di Leray. Si deducono alcune proprietà: per esempio i gruppi $H^*_{(k)}(X, \mathbf{Z})$ sono di tipo finito per ogni poliedro compatto X.

1. INTRODUCTION

The Alexander-Spanier cohomology [1] of the space X is defined considering the complex of the Alexander-Spanier cochains modulo the subcomplex of " locally zero " cochains. A cochain ω of order n is " locally zero " iff it vanishes on a neighbourhood of the diagonal $\nabla^k(X) = \{(x, \cdots, x) \mid x \in X\} \subset X^n$.

In this paper we define and study a new cohomology functor considering as " locally zero " a cochain which vanishes on a neighbourhood of

$$\nabla^n_p(X) = \{(x^1, \cdots, x^n) \mid x^i \in X , \quad \text{card} \quad \{x^1, \cdots, x^n\} \leq p\} \subset X^n.$$

We prove that the resulting functor is homotopic for compact spaces, and we construct a spectral sequence which generalizes the Leray spectral sequence; we deduce a finitess theorem for the cohomology groups of compact polyhedra.

I thank prof. Albrecht Dold for useful conversations.

2. RECALL AND CONSTRUCTIONS

We begin with a short presentation of the Alexander-Spanier cohomology (the absolute case).

We adopt the notations from the tract by E. Spanier: " Algebraic Topology ".

Let G be an R-module and X a topological space; we denote $X^p = X \times \cdots \times X$ (p times).

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Let

$$C^q(X, G) = \{\varphi \mid \varphi : X^{q+1} \to G\}$$

and let

$$\delta : C^q(X, G) \to C^{q+1}(X, G) \tag{1}$$

be defined by the formula:

$$\delta\varphi(x_0, x_1, \cdots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{q+1}). \tag{2}$$

$C^q(X,G)$ is a G-module in a natural manner and $C^*(X,G) = \{C^q(X,G), \delta\}_q$ is a cochain complex.

Let $C_0^*(X, G) \subset C^*(X, G)$ be the subcomplex of "locally zero" cochains; by definition, the cochain $\varphi \in C^q(X, G)$ is locally zero iff there is an open covering $\mathcal{U}$ of X such that

$$\varphi / \mathcal{U}^{q+1} = 0 \quad , \quad \mathcal{U}^{q+1} = \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha^{q+1} \subset X^{q+1}. \tag{3}$$

The cohomology complex $C^*(X, G)$ is acyclic, but the complex

$$\bar{C}^*(X, G) = C^*(X, G) / C_0^*(X, G), \tag{4}$$

generally, is not acyclic.

By definition, the homology of the complex $\bar{C}^*(X, G)$ is the "Alexander-Spanier cohomology of the space X with coefficients in G".

There is an alternative definition of the complex $\bar{C}^*(X, G)$.

If $\mathcal{U}$ is an open covering of X, let $X(\mathcal{U})$ denote the abstract simplicial complex whose q-simplexes are the points of the space $\mathcal{U}^{q+1}$. Let $C^*(\mathcal{U}, G)$ be the cochain complex of the simplicial complex $X(\mathcal{U})$ with coefficients in G.

If $\mathcal{U} < \mathcal{V}$ (the covering $\mathcal{V}$ is a refinement of the covering $\mathcal{U}$) then the restriction map

$$\psi_{\mathcal{U}}^{\mathcal{V}} : C^*(\mathcal{U}, G) \to C^*(\mathcal{V}, G) \tag{5}$$

defines a direct system $\{C^*(\mathcal{U}, G), \psi_{\mathcal{U}}^{\mathcal{V}}\}$.

We have

$$\varinjlim \{C^*(\mathcal{U}, G), \psi_{\mathcal{U}}^{\mathcal{V}}\} \sim \bar{C}^*(X, G). \tag{6}$$

We define now a new cohomology.

Let $p \in \mathbf{N}$ be a fixed number.

Let $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ be an open covering of X and let be

$$\mathcal{U}_{(p)}^{q+1} = \bigcup_{(\alpha_0, \alpha_1, \cdots, \alpha_q)} U_{\alpha_0} \times U_{\alpha_1} \times \cdots \times U_{\alpha_q} \tag{7}$$

such that the set $(\alpha_0, \alpha_1, \cdots, \alpha_q)$ contains $\leq p$ distinct elements.

We define:

$$X_r^{(p)}(\mathcal{U}, G) = \{a \mid a = \sum_{\text{finite}} \rho_i \xi_i , \rho_i \in G , \xi_i \in \mathcal{U}_{(p)}^{r+1}\}$$

$$C_{(p)}^q(\mathcal{U}, G) = \{\varphi \mid \varphi : \mathcal{U}_{(p)}^{q+1} \to G\} .$$

It is clear that:

$$C_{(1)}^q(\mathcal{U}, G) = C^q(\mathcal{U}, G) .$$

We define in a similar manner

(8) $$C_{(p)}^*(\mathcal{U}, G) = \{C_{(p)}^q(\mathcal{U}, G) , \delta\}_{q=0,1,\cdots}$$

and

(8′) $$\overline{C}_{(p)}^*(X, G) = \lim_{\overrightarrow{\mathcal{U}}} C_{(p)}^*(\mathcal{U}, G) .$$

We define

(9) $$H_{(p)}^*(X, G) = H_*(\overline{C}_{(p)}^*(X, G)) .$$

We have evidently:

$$H_{(1)}^*(X, G) = H^*(X, G) .$$

If $f \in Top(X, Y)$ and $\mathcal{U}$ is an open covering of Y then $f^{-1}\mathcal{U} = \{f^{-1}(U_\alpha)\}_\alpha$, $U_\alpha \in \mathcal{U}$, is an open covering of X and f defines a simplicial map:

$$f_* : X(f^{-1}\mathcal{U}) \to Y(\mathcal{U})$$

and also a morphism

$$f^* : C_{(p)}^*(\mathcal{U}, G) \to C_{(p)}^*(f^{-1}\mathcal{U}, G)$$

which commutes with the restriction morphisms: $\psi_{\mathcal{U}}^{\mathcal{V}}, \psi_{f^{-1}\mathcal{U}}^{f^{-1}\mathcal{V}}$; hence f defines a homomorphism:

$$f^* : H_{(p)}^*(Y, G) \to H_{(p)}^*(X, G) .$$

PROPOSITION 2.1. *Let* X, Y $\in$ Top, X *compact.*

Let $\Delta : X \times I \to Y$ *be a homotopy; let*

$$i_0 : X \to X \times \{0\} \hookrightarrow X \times I$$

$$i_1 : X \to X \times \{1\} \hookrightarrow X \times I$$

be the natural injections.

Then we have

$$(\Delta \circ i_0)^* = (\Delta \circ i_1)^* : H_{(p)}^*(Y, G) \to H_{(p)}^*(X, G) .$$

(*In other words if* $f_1 \simeq f_2 : X \to Y$, X *compact, then* $f_1^* = f_2^*$).

Proof. It is sufficient to prove

$$i_0^* = i_1^* \tag{10}$$

because $(\Delta \circ i_0)^* = i_0^* \circ \Delta^*$ and $(\Delta \circ i_1)^* = i_1^* \circ \Delta^*$.

By the fact that $\varinjlim$ commutes with the homology functor defined on the category of cochain-complexes, it follows that it is sufficient to prove that for any open covering $\mathcal{U}$ of $X \times I$ there is an open covering $\mathcal{V}$ of X finer than $(i_0^{-1} \mathcal{U}) \cap (i_1^{-1} \mathcal{U})$ and an homotopy:

$$k_r : X_r^{(p)} (\mathcal{V}, G) \to Y_{r+1}^{(p)} (\mathcal{U}, G) , \qquad r = 0 , 1 , \cdots$$

which connects $(i_0)_*$ and $(i_1)_*$.

If $\mathcal{S}$ is an open covering of X and $\mathcal{R}_n$ is the covering of I by the sets $\left(\frac{m}{n} , \frac{m+2}{n}\right)$, $0 \leq m \leq n-2$, then $\mathcal{S} \times \mathcal{R}_n$ is a covering of $X \times I$; X being compact the set of the coverings $\mathcal{S} \times \mathcal{R}_n$ is cofinal in the set of open coverings of $X \times I$.

Then it is sufficient to consider the case $\mathcal{U} = \mathcal{S} \times \mathcal{R}_n$; we consider on X the covering $\mathcal{V} = \mathcal{S}$.

Now we define the homotopy k_r by the formula:

$$k_r [x_0 , \cdots , x_r] =$$

$$= \sum_{i=0}^{r} (-1)^i \left[(x_0 , 0) , \cdots , (x_i , 0) , \left(x_i , \frac{1}{n}\right) , \left(x_{i+1} , \frac{1}{n}\right) , \cdots , \left(x_r , \frac{1}{n}\right)\right] +$$

$$+ \sum_{i=0}^{r} (-1)^i \left[\left(x_0 , \frac{1}{n}\right) , \cdots , \left(x_i , \frac{1}{n}\right) , \left(x_i , \frac{2}{n}\right) , \left(x_{i+1} , \frac{2}{n}\right) , \cdots , \left(x_i , \frac{2}{n}\right)\right] +$$

$$+ \cdots\cdots\cdots$$

$$+ \sum_{i=0}^{r} (-1)^i \left[\left(x_0 , \frac{n-1}{n}\right) , \cdots , \left(x_i , \frac{n-1}{n}\right) , \left(x_i , \frac{n}{n}\right) , \left(x_{i+1} , \frac{n}{n}\right) , \cdots , \left(x_r , \frac{n}{n}\right)\right]$$

for $[x_0 , \cdots , x_r] \in V_{(p)}^{r+1}$.

Remark 2.2. Probably the Proposition remains valid for any X, not necessary compact.

Example 2.3. Let X be a discrete space. Then, if we work with skew-symmetric cochains, which form a cochain-homotopy equivalent complex with defined complex $\overline{C}_p^* (X , G)$, we deduce:

$$H_{(p)}^* (X , G) = H^* (\mathcal{X} , G)$$

where $\mathcal{X}$ is the polyeder in which any vertex is a point of X and any $r + 1$ distinct points of X, $r \leq p$, form a r-simplex.

For example if $p = 2$ and X contains three points

$$H^*_{(2)}(X, G) \cong H^*(S^1, G).$$

3. Some properties of the functor $H^*_{(p)}(-, G)$

3.1. Proposition 3.1. *For any* r, $1 \leq r \leq p - 2$, *and any* X, $H^r_{(p)}(X, G) = 0$.

Proof. Let $\zeta \in X$ be a fixed point in X and $\eta : G \to \overline{C}^0_{(p)}(X, G)$ the augmentation

$$(\eta(g))(x) = g \quad , \quad g \in G \quad , \quad x \in X. \tag{12}$$

Let

$$k^r : \overline{C}^r_{(p)}(X, G) \to \overline{C}^{r-1}_{(p)}(X, G), \qquad 1 \leq r \leq p - 1,$$

be defined by the formula

$$(k^r \varphi)[x_0, \cdots, x_{r-1}] = \varphi([\zeta, x_0, x_1, \cdots, x_{r-1}]) \tag{13}$$

for $\varphi \in \overline{C}^r_{(p)}(X, G)$.

Then the relation holds:

$$(\delta_{r-1} k_r + k_{r+1} \delta_r)\varphi = \begin{cases} \varphi & \text{for } 1 \leq r \leq p-2, \varphi \in \overline{C}^r_{(p)}(X, G) \\ \varphi - \eta(\varphi(\zeta)), \varphi \in \overline{C}^0_{(p)}(X, G). \end{cases} \tag{14}$$

and hence the desired result.

The upper homotopy is used in the literature for proving the acyclicity of the cochain complex $C^*(X, G)$ (c.f. E. Spanier, *Algebraic Topology*, p. 307).

3.2. We indicate now a method which permits the evaluation of the groups $H^r_{(p)}(X, G)$ for any r, $0 \leq r \leq N$, N being a fixed natural number, arbitrarily choosed.

Hence, let N be an arbitrarily fixed natural number.

Definition 3.2. We say that the covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of the space X is a "N-covering" if and only if $\mathcal{U}^N = \{U^N_\alpha\}_\alpha$ is a covering of X^N.

If $\mathcal{U}$ is a N-covering of X and $N' \leq N$, then $\mathcal{U}$ is a N′-covering of X.

We introduce on Λ an arbitrary total order.

We define the n-nerve of the covering $\mathcal{U}$, and we write $\mathcal{N}_n(\mathcal{U})$, as the set:

$$\mathcal{N}_n(\mathcal{U}) = \left\{(\alpha_0, \alpha_1, \cdots, \alpha_n) \mid \alpha_0 < \alpha_1 < \cdots < \alpha_n, \alpha_i \in \Lambda, \bigcap_{1=0}^{n} U_{\alpha_i} \neq \varnothing \right\}. \tag{15}$$

If

$$\sigma = (\alpha_0 , \alpha_1 , \cdots , \alpha_n) \in \mathcal{N}_n (\mathcal{U})$$

let

$$U_\sigma = \bigcap_{i=0}^{n} U_{\alpha_i} .$$

Let $X^m_{\mathcal{U}} = \bigcup_{\alpha \in \Lambda} U^m_\alpha \subset X^m$ and $X^*_{\mathcal{U}} = \bigvee_{m=1}^{\infty} X^m_{\mathcal{U}}$.

Clearly,

$$\mathcal{U}^m = \{U^m_\alpha\}_{\alpha \in \Lambda}$$

is a covering of $X^m_{\mathcal{U}}$, and

$$\mathcal{N}_n (\mathcal{U}^m) = \mathcal{N}_n (\mathcal{U}) .$$

We suppose now $\mathcal{U}$ is a N-covering of X.

Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ be an open covering of X.

Let:

$$\mathcal{V}^r_{(p)} (\mathcal{U}) = \mathcal{V}^r_{(p)} \cap X^r_{\mathcal{U}} \subset X^r.$$

We denote

$$\overline{C}^r_{(p)\mathcal{U}} (\mathcal{V}, G) = \{ \varphi \mid \varphi : \mathcal{V}^{r+1}_{(p)} (\mathcal{U}) \to G \}$$

and

$$\overline{C}^r_{(p)\mathcal{U}} (X , G) = \varinjlim_{\mathcal{V}} \overline{C}^r_{(p)\mathcal{U}} (\mathcal{V}, G) . \tag{16}$$

For $r \leq N$ we have

$$\overline{C}^r_{(p)\mathcal{U}} (\mathcal{V}, G) = \overline{C}^r_{(p)} (\mathcal{V}, G) \qquad \text{and} \quad \overline{C}^r_{(p)\mathcal{U}} (X , G) = \overline{C}^r_{(p)} (X , G) .$$

Let

$$i^r_{\mathcal{V}} : \overline{C}^r_{(p)\mathcal{U}} (\mathcal{V} , G) \to \prod_{\sigma \in \mathcal{N}_0(\mathcal{U})} \overline{C}^r_{(p)} (U_\sigma \cap V, G)$$

be defined as follows:

for

$$\varphi \in \overline{C}^r_{(p)\mathcal{U}} (\mathcal{V}, G) , (i^r \varphi) = \{\varphi_\alpha\}_{\alpha \in \mathcal{N}_0(\mathcal{U})}$$

where

$$\varphi_\alpha = \varphi \mid U^{r+1}_\alpha \cap \mathcal{V}^{r+1}_{(p)} (\mathcal{U}) .$$

Let

$$d^r_{q,\mathcal{V}} : \prod_{\sigma \in \mathcal{N}_p(\mathcal{U})} \overline{C}^r_{(p)\mathcal{U}} (\mathcal{V} \cap U_\sigma , G) \to \prod_{\sigma \in \mathcal{N}_{q+1}(\mathcal{U})} \overline{C}^r_{(p)\mathcal{U}} (\mathcal{V} \cap U_\sigma , G)$$

be defined as follows: for any $\varphi = \{\varphi_\sigma\}_{\sigma \in \mathcal{N}_q(\mathcal{U})}$,

$$d^r_{q,\mathcal{V}} (\varphi) = \{\psi_{\overline{\sigma}}\}_{\overline{\sigma} \in \mathcal{N}_{q+1}(\mathcal{U})} (\mathcal{U})$$

$$\psi (\alpha_0 , \alpha_1 , \cdots , \alpha_q , \alpha_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi (\alpha_0 , \cdots , \hat{\alpha}_i , \cdots , \alpha_{q+1}) .$$

Let:

$$i^r = \lim_{\overrightarrow{\mathcal{V}}} i^r_{\mathcal{V}} : \overline{C}^r_{(p)\mathcal{U}}(X, G) \to \prod_{\sigma \in \mathcal{N}_0(\mathcal{U})} \overline{C}^r_{(p)\mathcal{U}}(U_\sigma, G)$$

$$d^r_q = \lim_{\overrightarrow{\mathcal{V}}} d^r_{q,\mathcal{V}} : \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^r_{(p)}(U_\sigma, G) \to \prod_{\sigma \in \mathcal{N}_{q+1}(\mathcal{U})} \overline{C}^r_{(p)}(U_\sigma, G) .$$

LEMMA 3.3. *If the space* X *is paracompact, then the sequence*:

$$S_{\mathcal{U}}: \quad \begin{aligned} &O \to \overline{C}^*_{(p)\mathcal{U}}(X, G) \xrightarrow{i^*} \prod_{\sigma \in \mathcal{N}_0(\mathcal{U})} \overline{C}^*_{(p)\mathcal{U}}(U_\sigma, G) \to \cdots \\ &\cdots \to \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)\mathcal{U}}(U_\sigma, G) \xrightarrow{d^*_p} \prod_{\sigma \in \mathcal{N}_{q+1}(\mathcal{U})} C^*_{(p)\mathcal{U}}(U_\sigma, G) \to \cdots \end{aligned}$$

is exact.

Proof. We prove that for any covering $\mathcal{V} = \{V_a\}_{a \in A}$ of the space X, the sequence $S_{\mathcal{U}}(\mathcal{V})$

$$S_{\mathcal{U}}(\mathcal{V}): \quad \begin{aligned} &O \to \overline{C}^*_{(p)\mathcal{U}}(\mathcal{V}, G) \xrightarrow{i^*_{\mathcal{V}}} \prod_{\sigma \in \mathcal{N}_0(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G) \to \cdots \\ &\cdots \to \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G) \xrightarrow{d^*_{q,\mathcal{V}}} \prod_{\sigma \in \mathcal{N}_{q+1}(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G) \to \cdots \end{aligned}$$

is exact.

If the sequence $S_{\mathcal{U}}(\mathcal{V})$ is exact for any $\mathcal{V}$, then also the sequence $S_{\mathcal{U}} = \lim_{\overrightarrow{\mathcal{V}}} S_{\mathcal{U}}(V)$ is exact.

We will interpret the elements of the groups $\overline{C}^*_{(p)\mathcal{U}}(\mathcal{V}, G)$, resp. $\prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G)$ as sections in a certain system of sheaves:

$$\underline{\overline{C}^*_{(p)\mathcal{U}}(\mathcal{V}, G)}, \qquad \text{resp.} \quad \underline{\prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)\mathcal{U}}(U_\sigma \cap \mathcal{V}, G)} \qquad \text{on} \quad X^*.$$

If $\varphi \in \overline{C}^*_{(p)\mathcal{U}}(\mathcal{V}, G)$, resp. $\prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G)$, we shall prolungate the function φ by O on $X^* - X^*_{\mathcal{U}}$, resp. $X^* - \mathcal{V}^*_{(p)}(\mathcal{U})$, and we obtain a function $\overline{\varphi}$ on X^*. The germs of the functions $\overline{\varphi}$ form a sheaf on X^*, which we denote by $\underline{\overline{C}^*_{(p)\mathcal{U}}(\mathcal{V}, G)}$, resp. $\underline{\prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^*_{(p)}(U_\sigma \cap \mathcal{V}, G)}$; all these sheaves are fine.

As the restrictions of the upper sheaves are also fine, the sequence $\mathcal{S}_{\mathcal{U}}(\mathcal{V})$ is exact.

The Lemma is proved.

COROLLARY 3.4. *If* X *is paracompact and* $\mathcal{U}$ *is a* N*-covering of* X, *then the sequence*:

$$(17) \quad \begin{aligned} &O \to \overline{C}^r_{(p)}(X, G) \xrightarrow{i^r} \prod_{\sigma \in \mathcal{N}_0(\mathcal{U})} \overline{C}^r_{(p)} \to \cdots \\ &\cdots \to \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^r_{(p)}(U_\sigma, G) \xrightarrow{d^r_q} \prod_{\sigma \in \mathcal{N}_{q+1}(\mathcal{U})} \overline{C}^r_{(p)}(U_\sigma, G) \to \cdots \end{aligned}$$

is exact for $r \leq N$.

Remarks 3.5. If $p = 1$ and A contains two elements, then the sequence $\mathcal{S}_{\mathcal{U}}$ is the short exact sequence which defines the Mayer-Vietoris sequence associated to the covering.

THEOREM 3.6. *If* X *is paracompact and* $\mathcal{U}$ *is a* N-*covering of* X *then there exists a spectral sequence in the first quadrant in which*:

$$E_1^{q,r} = \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} H^r_{(p)}(U_\sigma, G)$$

which converges to $H^r_{(p)}(X, G)$, *for* $r \leq N$.

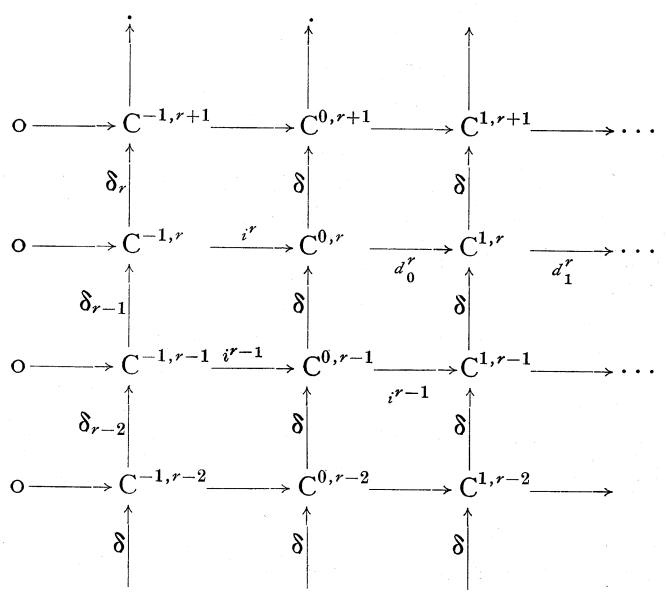
Proof. Let:

$$C^{-1,r} = \overline{C}^r_{(p)\mathcal{U}}(X, G), \qquad r = 0, 1, \cdots$$

$$C^{q,r} = \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} \overline{C}^r_{(p)}(U_\sigma, G), \qquad q, r = 0, 1, \cdots$$

Then the exact sequence $\mathcal{S}_{\mathcal{U}}$ can be written:

(C_{-1})



The diagram (C_{-1}) is commutative; all columns are complexes, and all lines are exact. The homology of the first column is the homology which we wish to evaluate. The homologies of the last columns are $\prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} H^*_{(p)}(U_\sigma, G)$.

We can modify the signs of the homomorphisms δ_r, in such a way that (C_{-1}) becomes a bicomplex.

If we consider the spectral sequence associated to the first filtration of the bicomplex in the first quadrant (C_0):

$$
(C_0) \qquad
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & C^{0,r+1} & \longrightarrow & C^{1,r+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & C^{0,r} & \longrightarrow & C^{1,r} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & C^{0,r-1} & \longrightarrow & C^{1,r-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}
$$

we obtain:

$$
{}'E_1^{q,r} = \begin{cases} C^{-1,r} & \text{for } q = 0 \\ 0 & \text{for } q > 0 \end{cases}
$$

and

$$
{}'E_2^{q,r} = \begin{cases} H_r(\{C^{-1,*}, \delta\}) & \text{for } q = 0 \\ 0 & \text{for } q > 0 ; \end{cases}
$$

hence the diagonal complex associated to the bicomplex (C_0) has the homology $H_*(\{C^{-1,*}, \delta\})$, the spectral sequence being degenerate.

If we consider the spectral sequence associated to the second filtration in the same bicomplex (C_0), we have:

$$
{}''E_1^{q,r} = \prod_{\sigma \in \mathcal{N}_q(\mathcal{U})} H_{(p)}^r (U_\sigma , G) ,
$$

and this last spectral sequence converges to the cohomology of the diagonal complex, hence the desired cohomology.

THEOREM 3.7. *If* $\mathcal{U}$ *is a finite* N-*covering of the space* X *and if* $H_{(p)}^r(U_\sigma , G)$ *are finite generated groups, for* p *fixed, then* $H_{(p)}^r(X , G)$, $r \leq N$, *are finite generated groups.*

THEOREM 3.8. *If* X *is a compact polyedron, and* G *is a finite generated* R-*module* (R *being a Notherian ring*), *then* $H_{(p)}^r(X , G)$ *are finite generated* R-*modules for any* r *and* p.

Proof. By the Theorem 3.7, it is sufficient to prove that for any N there exists a finite N-covering $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ of X such that U_σ, $(\sigma \in \mathcal{N}_q(\mathcal{U}))$, has the homotopy type of a finite discrete space. Now we shall construct a such covering.

We embed linearly X in an Euclidean space and we consider on X the induced metric d. Then there exists a positive number $0 < \theta$, such that any sphere in X of radius $\leq \theta$ is contractible and any finite intersection of such spheres has the homotopy type of a finite space. Now the Theorem will follow from the

LEMMA 3.9. *Let* (X, d) *be a metric space, and* S_i, $1 \leq i \leq N$, *spheres in* X, (*radius* $S_i) \leq r$. *Then there exist the spheres* $\Sigma_1, \cdots, \Sigma_n$, $n \leq N$, (*radius* $\Sigma_i) \leq 2^N \cdot r$ *such that*

$$\text{i)} \quad \bigcup_{i=1}^{n} \Sigma_i \supset \bigcup_{i=1}^{N} S_i$$

$$\text{ii)} \quad \Sigma_i \cap \Sigma_j = \varnothing , \qquad 1 \leq i < j \leq n.$$

Proof. We prove the Lemma by increased induction of N. If $N = 1$, the Lemma is clearly true. We suppose now the Lemma has been proved for $N \leq \overline{N}$ and we prove it for $N = \overline{N} + 1$. If $S_i \cap S_j = \varnothing$ for any $1 \leq i, j \leq \overline{N} + 1$, we take $\Sigma_i = S_i$, $1 \leq i \leq \overline{N} + 1$.

If there exist two distinct indices $1 \leq i, j \leq \overline{N} + 1$ (we suppose $i = 1, j = 2$), such that $S_1 \cap S_2 \neq \varnothing$, we can cover $S_1 \cup S_2$ by a sphere S of radius $\leq 2r$, and hence $\bigcup_{1 \leq i \leq \overline{N+1}} S_i$ can be covered by $\overline{N}$ spheres $S, S_3, S_4, \cdots, S_{\overline{N}+1}$ of radius $\leq 2r$, and by the inductive hypothesis the Lemma is proved.

Let be $r > 0$ such that $2^N r < \theta$. There exists a finite covering $\mathscr{V} = (V_\alpha)'_{\alpha \in \Lambda}$ of X by spheres of radius $\leq r$. Then $\mathscr{W}$ with

$$\mathscr{W} = (W_a)_{a \in A} \quad , \quad W_a = \bigcup_{1 \leq i \leq N} V_{\alpha_i} \quad , \quad \alpha_i \in \Lambda \ , \quad a = (\alpha_1, \cdots, \alpha_N)$$

is a N-covering of X.

By the Lemma 3.9 any W_a can be covered by a finite set of one to one disjoint spheres of radius $< \theta$, and so the Theorem is proved.

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