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Sandro Buoncristiano

Elementary proofs of some results of engulfing theory

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RENDICONTI

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SEZIONE I
(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Elementary proofs of some results of engulfing theory.* Nota di SANDRO BUONCRISTIANO, presentata (*) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Il presente lavoro è una continuazione di [2]. Lí si era stabilito un teorema di « Engulfing » (= inghiottimento) usando la teoria dei manici. Qui faremo vedere come i ben noti teoremi di Engulfing di Zeeman, Stallings e Bing si possono dedurre dal metodo descritto in [2]. Dimostreremo anche un teorema riguardante l'inghiottimento di un poliedro contenuto nel bordo di una varietà.

§ 0. INTRODUCTION

We place ourselves in the PL category (polyhedra and PL maps). Let X be a compact polyhedron in a manifold V. There are two basic definitions of Engulfing.

(S) given an open subset U of V we say that X can be *engulfed into* U if there exists an ambient isotopy (= continuous family of homeomorphisms) of V which carries X into U;

(Z) given a closed subpolyhedron C of V we say that X can be *engulfed from* C if X is contained in a regular neighbourhood of C in V (i.e. a compact neighbourhood of C which is a manifold and collapses to C).

(S) and (Z) are known as Stalling's and Zeeman's Engulfings respectively.

The Engulfing problem consists in finding sufficient conditions under which it is possible to engulf X (into U or from C accordingly). Engulfing

(*) Nella seduta del 12 aprile 1975.

is one of the most useful tools in PL topology and has been used successfully by Zeeman and Stallings in their proof of the generalised Poincaré Conjecture.

The main classical results are the following.

(ST) THEOREM (Stallings). *Let* V^v *be a manifold without boundary*, $U \subset V$ *an open subset*, $X^x \subset V$ *a compact polyhedron and suppose that* (V, U) *is at least x-connected and* $x \leq v - 3$. *Then* X *can be engulfed into* U.

(ZT) THEOREM (Zeeman). *Let* X^x, C^c *be subpolyhedra of the compact manifold* V, C *closed and* X *compact*, $X \subset V - \partial V$, *and suppose the following hypotheses are satisfied*:

(V, C) *is k-connected*, $k \geq 0$ *and there exists a homotopy of* X *into* C *which is modulo* C;

$$x \leq v - 3 \quad ; \quad c \leq v - 3 \quad ; \quad c + x \leq v + k - 2 \quad ; \quad 2x \leq v + k - 2 .$$

Then X is contained in a regular neighbourhood of C in V.

There is also a third type of Engulfing, called Radial Engulfing, which says roughly the following:

(B) Given U (or C) as in (S), (Z) when can X be engulfed into U (or from C) in such a way that the engulfing isotopy moves each point of V along " a prescribed direction "? (see theorem below for a precise statement).

This Engulfing has been considered by Bing in [1]. His result is the following

(BT) THEOREM (Bing). *Let* $\{A_\alpha\}$ *be a collection of sets in a manifold* M^n *without boundary*, $X^x \subset M$ *a compact subpolyhedron* $x \leq n - 3$, U *an open subset of* M.

Suppose that for each compact y-dimensional polyhedron Y, $y \leq x$, there exists a homotopy $H : Y \times [0, 1] \to M$ such that

(1) $H_0 = \mathrm{id}$; $H_1(Y) \subset U$;

(2) for each point $y \in Y$, $H(y \times [0, 1])$ lies in one element of $\{A_\alpha\}$.

Then, for each $\varepsilon > 0$, there is an engulfing isotopy $H : M \times [0, 1] \to M$ such that $H_0 = \mathrm{id}$, $H_1(X) \subset U$ and, for each point $p \in M$, there are $x + 1$ elements of $\{A_\alpha\}$ such that the track $H(p \times [0, 1])$ lies in the ε-neighbourhood of, the sum of these $x + 1$ elements.

The proofs of ST, ZT and BT are long and technical.

In [2] we considered Engulfing from a slightly different point of view, namely:

(RS) given X^x in a cobordism $(W, \partial_- W, \partial_+ W)$ we say that X can be *engulfed from* $\partial - W$ if X is contained in a collar of $\partial - W$ in W.

The above definition is labelled RS because it can be found in Rourke and Sanderson [3].

Our main result of [2] was the following.

(A) *Engulfing Theorem.* Let W^w be a compact cobordism, $X^x \subset W$ a compact subpolyhedron, $X \cap \partial_+ W = \varnothing$ and suppose that the following hypotheses are satisfied:

$$X \text{ is } \partial_- W\text{-inessential}$$

$$(W, \partial - W) \text{ is } k\text{-connected}$$

$$2x \leq w + k - 2 \quad ; \quad x \leq w - 3 .$$

Then X can be engulfed from $\partial_- W$.

The proof is very short and elementary and uses only general position, handle-theory and induction.

We also stated, without proof, three corollaries of Theorem A, corresponding to the well-known versions of Engulfing described above.

The purpose of the present paper is to give a short proof of these corollaries, which form Theorems 2, 4, 7 below.

There are two remarks to be made at this point. The first is that Theorem 2 improves on ZT because it does not assume the homotopy of X to be modulo any subset of C. The second is that we do not derive Theorems 2, 4, 7 as direct corollaries of (A); but each of the theorems is proved independently of (A) although the proof is based on the same handle-theory arguments used to establish (A).

We also look at the engulfing of a polyhedron X contained in the boundary of a manifold W and show that, under reasonable hypotheses, this engulfing is possible even if X is in codimension two in ∂W (Theorem 8).

As this paper is a continuation of [2] we refer the reader to that paper for all the notation and terminology. We only recall a definition: a cobordism with boundary is a compact manifold W^w together with two disjoint $(w-1)$-dimensional submanifolds $\partial_- W$, $\partial_+ W$ of ∂W. We set $M = \mathrm{cl}(\partial W - \partial_- W - \partial_+ W)$.

§ 1. ENGULFING THEOREMS

We shall need the following addendum to Theorem [2], 1.

1. RELATIVE ENGULFING THEOREM. *The conclusion of Theorem 1 in* [2] *remains true if*: $(W, \partial_- W, \partial_+ W)$ *is a cobordism with boundary*, $M \cong \partial_- W \times [0, 1]$ *and* $X \cap (\partial W - \partial_- W) = \varnothing$.

Proof. Let W_0 be a collar on $\partial_- W$ extending a given collar-structure on M and consider a nice handle-decomposition of W on W_0. All the arguments used in the proof of the theorem remain valid in this relative situation and all the isotopies may be taken to be modulo ∂W. Hence the addendum follows.

2. THEOREM (Engulfing à la Zeeman). *Let* C^c, X^x *be closed subpolyhedra of the compact manifold* V^w, *with* X *compact and satisfying the following hypotheses*:

$$X \cap \partial V = \varnothing \quad ; \quad x \leq w - 3 \quad , \quad c \leq w - 3$$
$$c + x \leq w + k - 2$$
$$2x \leq w + k - 2$$

the pair (V, C) *is k-connected and there is a homotopy of* X *into* C. *Then* X *is contained in a regular neighbourhood of* C *in* W.

Proof. First we deal with the case $C \cap \partial V = \varnothing$. Let N be a regular neighbourhood of C in V, with $N \cap \partial V = \varnothing$, and consider the cobordism $(W, \partial_- W, \partial_+ W) = (V - \operatorname{Int} N, \partial N, \partial V)$.

Step 1. $(W, \partial_- W)$ is k-connected.

Proof of Step 1. Consider the excision map $(W, \partial_- W) \overset{j}{\hookrightarrow} (V, N)$, and the induced map $j_1 : \pi_1(W, \partial_- W) \to \pi_1(V, N)$ of the corresponding fundamental groups. We claim that j_1 is an isomorphism. This follows immediately from observing that the track of a homotopy of a relative loop in V, N has dimension ≤ 2, so that, by general position, it can be isotoped off C and hence off N, because N is a mapping cylinder on C. Therefore Step 1 follows from Hurewicz's theorem.

Step 2. We shall prove the following: suppose $V = W' \cup H$, $N \subset W'$, index $H \geq k + 1$. Then there exists a regular neighbourhood, N', of C in V and a homotopy, $f' : X \times [-1, 1] \to V$, of X into W', such that f' is modulo N' and $N' \subset N$.

Proof of Step 2. The important part of this step is to obtain that the homotopy be modulo N' for some N'. Note that, if $f : X \times [0, 1] \to V$ is the given homotopy of X into C, then f obviously throws X into W'; however we cannot take $f = f'$ and $N = N'$ because f may not be modulo N.

In order to construct f' we assume, first of all, f in general position and let $P \subset X \times [0, 1]$ be the subpolyhedron given by $f^{-1}(f(X \times [0, 1] \cap C))$. By general position

$$\dim f(X \times [0, 1] \cap C \leq x + 1 + c - w ,$$

hence $\dim P \leq x + 1 + c - w$ because f is non-degenerate (as usual, w.l.o.g. we assume $f(X \times [0, 1]) \cap \partial W = \varnothing$). Then the shadow, Ω_P, of P in $X \times [0, 1]$ has dimension $\leq x + 2 + c - w$. We now look at the dimension of $f(\Omega_P) \cap D$, where D = fibre of H. General position gives:

$$\dim (f(\Omega_P) \cap D) \leq (w - k - 1) + (x + 2 + c - w) - w \leq -1$$

because, by hypothesis, $c + x \leq w + k - 2$.

Therefore we can assume that $f(\Omega_P)$ does not meet D. Now let f^+, N, M, h, g be defined as in the Proof of [2] Theorem 1 (Step 1) after replacing Ω, W by Ω_P, V respectively. It is immediately seen that $g: X \times [-1, 1] \to V$ provides a homotopy which throws X off D and is modulo $C \cup J$ where J is a convenient neighbourhood of $X \cap C$ in X. Thus, because C is closed, there exists a regular neighbourhood N_0 of C such that $g: X \times [-1, 1] \to V$ is modulo N_0.

By construction the homotopy g pushes X off the fibre D. Then there is a handle-move which engulfs X into W'.

The composition of g with this handle-move provides a homotopy $X \times [-1, 1] \to V$ which carries X into W' and is modulo N_0. Hence we can take f' to be this homotopy and N' to be N_0.

This concludes the proof of Step 2.

Step 3. There exists a regular neighbourhood N' of C in V and a homotopy $f': X \times [-1, 1] \to V$ which throws X into a collar of $\partial_- W$ in W and is modulo N'.

Proof of Step 3. Choose a nice handle-decomposition of W on $\partial_- W$ and consider $W_{(k)} = \partial_- W \cup \{\text{handles of index} \leq k\}$. Combining Step 2 above with the inductive procedure indicated in the Proof of [2], Theorem 1 (Step 4), one easily deduces the existence of a homotopy $f'': X \times [-1, 1] \to V$ which carries X into $W_{(k)}$ and is modulo a convenient regular neighbourhood N' of C in V, $N' \subset \operatorname{Int} N$. We know that $W_{(k)}$ is a regular neighbourhood of $\partial_- W \cup k$ where k is a k-dimensional polyhedron. Then, because $(W, \partial_- W)$ is k-connected, there is a homotopy $f''': W_{(k)} \times [0, 1] \to V$ which carries $W_{(k)}$ into a collar of $\partial_- W$ and is modulo $N' \subset \operatorname{Int} N$. The composition of f'' and f''' provides the required homotopy f'.

Step 4. Completion of the proof of the theorem in the case $C \cap \partial V = \varnothing$. Consider the cobordism $(\overline{W}, \partial_- \overline{W}, \partial_+ \overline{W}) = (\mathrm{cl}(V - N'), \partial N', \partial V)$. In step 3 we have proved that $\overline{X}$ is $\partial_- \overline{W}$ inessential in $\overline{W}$ ($\overline{X} = X - \operatorname{Int} N'$). Therefore Theorem 1 of [2] applies to give an ambient isotopy of $\overline{W}$ which carries $\overline{X}$ into a collar W_0 of $\partial_- \overline{W}$ and is modulo $\partial_- \overline{W} \cup \partial_+ \overline{W}$.

Then we can extend this isotopy to an isotopy of the whole V by means of the identity on N'. As $N' \cup W_0$ is obviously a regular neighbourhood of C in W, the theorem is proved in the case $C \cap \partial V = \varnothing$. If, on the contrary, $C \cap \partial V \neq \varnothing$, then the proof is quite similar, the only difference being that the cobordism $(W, \partial_- W, \partial_+ W)$, defined by excising a regular neighbourhood of C in V, is now a cobordism with boundary, so that this case is a consequence of the relative engulfing theorem.

We omit the proof of the following addendum, which is easy.

3. *Addendum*. The conclusion of the above theorem continues to hold if X intersects the boundary ∂V and the given homotopy of X into C is modulo ∂V.

4. THEOREM (Engulfing à la Stallings). *Let* $(W^w, \partial_- W, \partial_+ W)$ *be a cobordism*, $U \supset \partial_- W$ *an open subset*, $X^x \subset W$ *a compact subpolyhedron*,

$X \cap \partial_+ W = \varnothing$. *Suppose*: (W, U) *is y-connected*, $y = 2x + 2 - w$, *and there is a homotopy* $f : X \times [0, 1] \to W$ *of* X *into* U, *which is modulo* $\partial_- W$. *Then there exists an isotopy of* W *carrying* X *into* U.

Proof. The proof goes by induction on the dimension of the polyhedron X, the induction starting trivially with $\dim X = -1$ or 0.

Suppose, then, we have proved the theorem for $\dim X < x$ and let us prove it for $\dim X = x$. Choose a regular neighbourhood N_1 of $f(X_1)$ in U, with $N_1 \cap \partial_+ W = \varnothing$, and consider the cobordism $(W', \partial_- W', \partial_+ W') = (\mathrm{cl}(W - N_1), \partial\,\mathrm{cl}(W - N_1) - \partial_+ W, \partial_+ W)$. Because X is in codimension at least three, one proves, as in Theorem 2 (Step 1), that (W', U') is y-connected where $U' = U \cap W'$. Consider, then, a handle decomposition of W' on (a collar of) $\partial_- W'$. We know that $W_{(y)}$ (handles of index $\leq y$) is a regular neighbourhood of $\partial_- W' \cup Y$, where Y is a y-dimensional polyhedron. Now it is immediately seen that $W', \partial_- W', \partial_+ W', U', Y$ satisfy the same hypotheses as $W, \partial_- W, \partial_+ W, U, X$ respectively in the statement of this theorem. Therefore, since $y < x$, we may assume, by induction, that there exists an ambient isotopy which engulfs Y into U' and is modulo $\partial_- W'$. Thus we can extend this isotopy to the whole of W by means of the identity on N_1 and so assume that $Y_1 \subset U \subset W$, where $Y_1 = \partial_- W \cup N_1 \cup Y$.

Now, from general-position arguments it follows at once that the pair (W, Y_1) is y-connected. Let C_x be the x-skelecton of Y_1 in a triangulation of W having Y_1 and $f(X_1)$ as subcomplexes. The quadruple $(W, \partial W, X, C_x)$ satisfies the same hypotheses as $(W, \partial W, X, C)$ in Theorem 2 and Addendum 3 above.

Therefore, given a regular neighbourhood N of C in W there exists an isotopy of W, which engulfs X into $N \subset U$ and the theorem is proved.

Remarks.

5. The conclusion of the above theorem remains true if $X \cap \partial_+ W \neq \varnothing$ and the homotopy f is modulo $\partial W = \partial_- W \cup \partial_+ W$.

6. The above theorem (and its proof) makes sense even for $\partial_- W = \varnothing$, in which case the statement reduces to that of [2], Corollary 3.

We now hint at that type of engulfing named 'radial engulfing' by Bing ([1]).

Let $(W, \partial_- W, \partial_+ W)$ be a cobordism, $\{S_j\}$ a collection of subpolyhedra of W, $\varepsilon > 0$ a real number, N_j the ε-neighbourhood of S_j in some fixed triangulation of W. If λ is a positive integer, a homotopy in W is defined to be λ-*radial* if the track of each point is contained in the union of at most λ elements of $\{N_j\}$; if a homotopy is λ-radial but not $(\lambda - 1)$-radial, then we say that λ is the *degree of radiality* of the homotopy.

We also say 'radial' instead of 1-radial. Then the notion of *radial k-connectedness* is the obvious one.

7. THEOREM (Engulfing à la Bing). *Let* $(W^w, \partial_- W, \partial_+ W)$ *be a cobordism,* $X^x \subset W$ *a subpolyhedron,* ε, $\{S_j\}$, $\{N_j\}$ *as above and suppose that:* $(W, \partial_- W)$ *is radially k-connected*;

there is a radial homotopy of X *into a collar of* $\partial_- W$, *the homotopy being modulo* $\partial_- W$;

$$2x \leq w + k - 2 \quad , \quad x \leq w - 3$$

Then, given a collar W_0 *on* $\partial_- W$ *in* W, *there exists an ambient isotopy, which is kw-radial and engulfs* X *into* W_0.

Idea of proof. First we choose a 'radial' handle-decomposition, i.e. one where each handle is contained in the interior of one element of $\{N_j\}$. Then the proof follows the same pattern as the non-radial case [2], Theorem 1 and it is very easy to check that each of the isotopies there constructed can now be assumed to have such a degree of radiality that the final isotopy of W is *kw*-radial. We leave the details to the reader.

The following theorem deals with the engulfing of a polyhedron X which is contained in the boundary of a high-dimensional manifold W^w $(w \geq 6)$. The result is obtained by combining the Proof of [2], Theorem 1 with the method of [3], Theorem 7.10 (1).

8. THEOREM. *Let* $(W^w, \partial_- W, \partial_+ W)$ *be a cobordism,* $X^x \subset \partial W$, *and suppose that*:

X is ∂_- W-*inessential*

$$2x \leq w + k - 3 \quad ; \quad x \leq w - 3 \quad , \quad w \geq 6 .$$

Then X *can be engulfed from* $\partial_- W$.

Proof. We distinguish the two cases: $x \leq w - 4$; $x = w - 3$. $x \leq w - 4$. To prove the theorem in this case it suffices to establish that X is ∂_-W-inessential by a homotopy which takes place in ∂W. Then one can proceed exactly in the same way as in the Proof of [2], Theorem 1, taking care to perform all constructions in ∂W: this is made possible by the assumption $2x \leq w + k - 3$.

We assume, w.l.o.g., $X^x \subset \partial_+ W$ and write $W = W' \cup H$ where W' is a cobordism on $\partial_- W$. Let $f : X \times [0, 1] \to W$ be the given homotopy. We shall replace f by a homotopy f'' which makes X ∂_-W inessential and takes place in $\mathrm{cl}(W' \cup H - W' \cap H)$; the result will then follow from induction.

By general position $f(X \times [0, 1]) \not\supset D$. Let q be a point of $D - f(X \times [0, 1])$. There is an obvious retraction (radial projection) of $H - q$ onto $g(\partial I^i \times I^{w-i} \cup I^i \times \partial I^{w-i})$, $g : I^i \times I^{w-i} \to H$ being the characteristic map of H. The composition of f with this retraction gives a new homotopy $f' : X \times [0, 1] \to W$ making X ∂_-W inessential. In general $f'(X \times [0, 1])$ will intersect the attaching tube $W' \cap H = g(S^{i-1} \times I^{w-i})$. Therefore we need a further modification. Choose a point $v \in I^{w-i}$ and consider the circle $C_v = g(S^{i-1} \times v)$. Because $x \leq w - 4$ general position in $\partial W'$ gives $f'(X \times [0, 1]) \cap C_v = \varnothing$.

Let r be the obvious deformation retraction of $W' \cup \partial H - C_v$ onto $\mathrm{cl}(W' \cup \partial H - W' \cap H)$. Then one sees immediately that $f'' = rf'$ is the required homotopy.

If $x = w - 3$. From $2x \leq w + k - 3$ it follows $k \geq w - 3$. Because $w \geq 6$ we can proceed as in [3], Theorem 7.10 (1), i.e. first we eliminate all handles of index $\leq w - 3$ by an ambient isotopy, then we shift X off the fibres of the remaining handles using general position and finally engulf by handle-moves.

The above theorem deals only with the case $w \geq 6$. If $w \leq 5$, it is not known whether it is possible to engulf a polyhedron $X \subset \partial W$ which has codimension two in ∂W. This problem is related to the well-known conjecture of Zeeman: there exists a compact 4-dimensional contractible manifold V^4 and an $S^1 \subset \partial V$ such that S^1 is essential in ∂V and S^1 does not bound a disk in V (see [5]). Certainly, if we assume X to have codimension three in ∂W, i.e. $w = 5$, $x = 1$, $k \geq 2$ or $w = 4$, $x = 0$, $k \geq 0$, then engulfing is possible by the proof of the above theorem.

References

[1] Bing R. H. (1967) – *Radial Engulfing*. Conference on Topology of Manifolds, Michigan State University.

[2] Buoncristiano S. (1975) – *Handle-theory and Engulfing*, Procedings of Cambridge Phil. Society, *78*, 111–116.

[3] Rourke C. P. and Sanderson N. J. (1972) – *Introduction to Piecewise-Linear Topology*, Springer-Verlag.

[4] Stallings T. (1962) – *The Piecewise-Linear Structure of Euclidian Space*, « Proc. Com. Phil. Soc. », *58*, 481–488.

[5] Zeeman E. C. – *Seminar on Combinatorial Topology (notes)*, I.H.E.S. (Paris) and University of Warwick (Coventry), 1963–66.