Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A remark on the differentiability for Green's operators of variational inequalities

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.4, p. 479–481.

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<http://www.bdim.eu/item?id=RLINA_1975_8_58_4_479_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Matematica. — *A remark on the differentiability for Green's operators of variational inequalities.* Nota di HUGO BEIRÃO DA VEIGA (*), presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — È stato dimostrato in [1] che l'operatore P definito da (3) è differenziabile nell'origine, inteso come operatore da $L^2(\Omega)$ in $L^2(\Omega)$. In questa Nota si osserva che continua a sussistere lo stesso risultato se P viene inteso come operatore da $L^2(\Omega)$ in $W^{1,2}(\Omega)$ ed inoltre come quest'ultimo possa essere ulteriormente generalizzato.

This Note is concerned with the recent paper [1] to which the reader is referred for terminology, notation and further details.

Let Ω be an open bounded set in the n-dimensional Euclidean space $\mathbf{R}^n$ and let Γ be the boundary of Ω. We assume that Ω and Γ are smooth.

We denote by $\| \ \|_p$ and $\| \ \|_{s,p}$ the usual norms in the space $L^p(\Omega)$ and $W^{s,p}(\Omega)$ respectively, and we put $H = L^2(\Omega)$, $\| \ \| = \| \ \|_2$. We shall consider also the spaces $L^p(\Gamma)$ and $W^{s,p}(\Gamma)$ with the usual norms $| \ |_p$ and $| \ |_{s,p}$ respectively.

Let now $\alpha : \mathbf{R} \to 2^{\mathbf{R}}$ and suppose that $0 \in \alpha(0)$; we say that the graph α is differentiable at the origin, with finite derivative α', if the following condition holds:

(1) for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $|z - \alpha' y| \leq \varepsilon |y|$, for all $z \in \alpha(y)$, for all $y \in]-\delta_\varepsilon, \delta_\varepsilon[\cap D(\alpha)$.

We say that α is differentiable at the origin with $\alpha' = +\infty$ if

(2) for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $|y| \leq \varepsilon |z|$, for all $z \in \alpha(y)$, for all $y \in]-\delta_\varepsilon, \delta_\varepsilon[\cap D(\alpha)$.

In the sequel β and γ are two maximal monote graphs on $\mathbf{R}$ verifying $0 \in \beta(0)$, $0 \in \gamma(0)$. It is well known that for every $u \in H$ there exists a unique function $Pu \in W^{2,2}(\Omega)$ satisfying

$$(3) \qquad \begin{cases} -\Delta Pu + \gamma(Pu) + Pu \ni u, & \text{a.e. in } \Omega \\ -(\partial Pu/\partial n) \in \beta(Pu), & \text{a.e. on } \Gamma, \end{cases}$$

where $\partial/\partial n$ is the outward normal derivative; moreover $\|Pu\|_{2,2} \leq c\|u\|$. We denote by c constants depending only on Ω, n, β and γ.

In [1] we have introduced a method that applies, in particular, to the study of the differentiability of the Green's operator P (1). More precisely

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(**) Nella seduta del 12 aprile 1975.

(1) In [1] we derive from this result a theorem on the bifurcation points for the operator P.

we have proved that:

THEOREM I: (i) *If* γ *is differentiable at the origin with* $\gamma' = +\infty$ *then the operator* P *is Fréchet differentiable and* $\mathrm{DP}(0) = 0$.

(ii) *If* β *and* γ *are differentiable at the origin with* $\gamma' < +\infty$ *and* $\beta' = +\infty$ *then the operator* P *is Fréchet differentiable at the origin and* $\mathrm{DP}(0) = \mathrm{A}$ *is the Green's operator for the linear Dirichlet problem*

$$\begin{cases} -\Delta \mathrm{A}u + \mathrm{A}u + \gamma' \mathrm{A}u = u & \textit{in} \quad \Omega, \\ \mathrm{A}u = 0 & \textit{on} \quad \Gamma. \end{cases} \tag{4}$$

(iii) *If* $\gamma' < +\infty$ *and* $\beta' < +\infty$ *then* $\mathrm{DP}(0) = \mathrm{A}$ *is the Green's operator for the linear problem*

$$\begin{cases} -\Delta \mathrm{A}u + \mathrm{A}u + \gamma' \mathrm{A}u = u & \text{in} \quad \Omega, \\ -\partial \mathrm{A}u/\partial n = \beta' \mathrm{A}u & \text{on} \quad \Gamma. \end{cases} \tag{5}$$

Obviously Theorem I is equivalent to prove that

$$\lim_{\|u\| \to 0} \frac{\|\mathrm{P}u - \mathrm{A}u\|}{\|u\|} = 0, \tag{6}$$

where $\mathrm{A} = 0$ in case (i).

It was remarked to the author (oral communication) by J. Hernandez that one can prove that

$$\lim_{\|u\| \to 0} \frac{\|\mathrm{P}u - \mathrm{A}u\|_{1,2}}{\|u\|} = 0. \tag{7}$$

The aim of this note is to verify that (7) is a trivial consequence of the estimates obtained in [1]. For brevity when we write " $\leq c\varepsilon \|u\|$ " it is understood that the corresponding estimate is true for $\|u\|$ sufficiently small.

Cases (ii) and (iii):

Put $\mathrm{R}u = \mathrm{P}u - \mathrm{A}u$. In [1] we succeed in proving that (cf. [1], (1.21), (1.22) and (1.25))

$$\|\Delta \mathrm{R}u - \mathrm{R}u - \gamma' \mathrm{R}u\|_q \leq c\varepsilon \|u\| \qquad \text{in cases (ii) and (iii)}, \tag{8}$$

with $q < 2$ (2), and we also show

$$|\mathrm{R}u|_2 \leq c\varepsilon \|u\| \qquad \text{in case (ii)}, \tag{9}$$

$$|\partial \mathrm{R}u/\partial n + \beta' \mathrm{R}u|_2 \leq c\varepsilon \|u\| \qquad \text{in case (iii)}. \tag{10}$$

(2) To prove (6) we had chosen in [1] $q = (2^*)'$ where $2^* = 2n/(n-2)$ is the Sobolev imbedding exponent of $\mathrm{W}^{1,2}(\Omega)$ and $1/(2^*)' = 1 - (1/2^*)$. To prove (7) we made the same choice of q. If $n \leq 2$ then $q < 2$ can be arbitrarly; but in this case the results can be strengthened.

From (8), (9), (10) and from well known estimates for solutions of linear equations we deduce immediatly, in our paper [1], relation (6). But from exactly the same estimates (8), (9), (10) one trivially derives relation (7). In fact multiplying $-\Delta \mathrm{R}u + \mathrm{R}u + \gamma' \mathrm{R}u$ by $\mathrm{R}u$, integrating in Ω and applying Green's formulae it follows that (we recall that $\gamma' \geq 0$)

$$\|\mathrm{R}u\|_{1,2}^2 \leq \int_\Gamma (\partial \mathrm{R}u/\partial n)\, \mathrm{R}u\, d\Gamma + \int_\Omega (-\Delta \mathrm{R}u + \mathrm{R}u + \gamma' \mathrm{R}u)\, \mathrm{R}u\, dx. \tag{11}$$

In case (ii) from the corresponding estimates (8), (9) it follows then that

$$\|\mathrm{R}u\|_{1,2} \leq c\varepsilon \|u\|, \tag{12}$$

because $\| \ \|_{2^*} \leq c \| \ \|_{1,2}$.

Analogously in case (iii) the corresponding estimates (8), (10) give (12) because we have in (11)

$$\int_\Gamma (\partial \mathrm{R}u/\partial n)\, \mathrm{R}u\, d\Gamma \leq \int_\Gamma (\partial \mathrm{R}u/\partial n + \beta' \mathrm{R}u)\, \mathrm{R}u\, d\Gamma.$$

Finally (i) follows from (6) (i.e. from $\|\mathrm{P}u\| \leq c\varepsilon \|u\|$) and from equation (3).

We remark that (7) can be further generalized. Consider for example case (iii). Formula (8) holds for all $q < 2$ (as proved in [1]) and consequently (8) holds with $\| \ \|_q$ replaced by $\| \ \|_{s,2}$, for all $s < 0$. From this estimate, from (10) and from known results for linear equations it follows that $\|\mathrm{R}u\|_{3/2,2} \leq c\varepsilon \|u\|$; consequently

$$\lim_{\|u\| \to 0} \frac{\|\mathrm{P}u - \mathrm{A}u\|_{3/2,2}}{\|u\|} = 0. \tag{13}$$

This relation can be generalized.

REFERENCES

[1] H. BEIRÃO DA VEIGA – *Differentiability for Green's operators of variational inequalities and applications to the calculus of bifurcation points* (to appear in thee «Journal of Functional Analysis».