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Linear operators on certain completion of the s-d-Rings over the integers. Nota II

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Algebra. — *Linear operators on certain completion of the s-d-Rings over the integers.* Nota II di ESAYAS GEORGE KUNDERT, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Quale applicazione della precedente Nota I con lo stesso titolo, si mostra come, completando l'*s–d*–anello sopra gli interi in modo opportuno, si ottiene un calcolo formale per sé interessante

PART II

To have less cumbersome notations we will carry out the following theory for the operators: $D, \bar{D}, S, \bar{S}, \sigma, \bar{\sigma}$ and the basis $\{x_i\}$ only. The reader should, however, keep in mind that these operators might always be replaced by the operators $D, \bar{D}, S_m, \bar{S}_m, \sigma_m, \sigma_{-m}$ and the basis $\{x_{mi}\}$ respectively. This will be obvious from the preceding development.

Let $\hat{\mathfrak{A}}$ be the completion of $\mathfrak{A}$ with respect to the inteal neighborhood system of the zero: $(x_0)^* \supset (x_1)^* \supset \cdots \supset (x_i)^* \supset \cdots$ (The topology defined on $\mathfrak{A}$ by this neighborhood system is Hausdorff since $\bigcap\limits_{i=0}^{\infty} (x_i)^* = \varnothing$). Each element $c \in \hat{\mathfrak{A}}$ may be represented as $c = \sum\limits_{k=0}^{\infty} \gamma_k x_k$ with $\gamma_k \in \mathbf{Z}$ and such a series is always an element of $\hat{\mathfrak{A}}$. We specifically single out the following elements of

$$\hat{\mathfrak{A}} : \bar{x}_{-i} = \sum_{k=0}^{\infty} \binom{k+i+1}{i-1} x_k , \qquad i = 1, 2, \cdots .$$

Note that these elements are not in $\mathfrak{A}$.

From the multiplication on $\hat{\mathfrak{A}}$ we obtain in particular the following products:

$$\text{(IV)} \quad \bar{x}_i \bar{x}_j = \sum_s \alpha_{ij}^s \bar{x}_s \quad \text{where} \quad \alpha_{ij}^s = (-1)^{i+j+s} \binom{j}{s-i} \binom{s}{j} \quad \text{for } i, j > 0$$

$$\alpha_{-m,n}^{-s} = \alpha_{m-1,n}^{n-1+s} \qquad \text{for } m, n > 0$$

$$\alpha_{-m,-n}^{-s} = \alpha_{m-1,n-1}^{m+n-s-1} \quad \text{for } m, n > 0.$$

From (IV) it follows that $\bar{x}_{-1} \bar{x}_j = \bar{x}_{-1}$ for all integers j. Therefore $\bar{x}_{-1} x_k = 0$ for $k > 0$ and therefore $\bar{x}_{-1} c = \sigma(c) \bar{x}_{-1}$. Let us introduce the linear operators $Q_i : Q_i(c) = \bar{x}_i c$ and we may put: $\bar{x}_{-1} c = Q_{-1} \sigma(c)$. Note: $\hat{\mathfrak{A}}$ cannot be an

(*) Nella seduta dell'8 marzo 1975.

integral domain! It is also easy to see that the $\{\bar{x}_i\}_{i=\text{integer}}$ do not form a basis of $\hat{\mathfrak{A}}$. The subspace $\tilde{\mathfrak{A}}$ spanned by the $\bar{x}$'s is however a (proper) subalgebra of $\hat{\mathfrak{A}}$.

Let us now look at linear operators on $\hat{\mathfrak{A}}$. We may at once extend the operators σ, D, S onto $\hat{\mathfrak{A}}$ by letting:

$$\sigma\left(\sum_{k=0}^{\infty}\gamma_k x_k\right) = \gamma_0 \quad , \quad \mathrm{D}\left(\sum_{k=0}^{\infty}\gamma_k x_k\right) = \sum_{k=1}^{\infty}\gamma_k x_{k-1}$$

and

$$\mathrm{S}\left(\sum_{k=0}^{\infty}\gamma_k x_k\right) = \sum_{k=0}^{\infty}\gamma_k x_{k+1} .$$

It is clear that σ is again an algebra homorphism from $\hat{\mathfrak{A}}$ onto $\mathbf{Z}$, D is a so-called weak semi-derivation satisfying properties (1) through (4) but not (5) and for S we have: $\mathrm{DS} = \mathrm{I}$ and $\mathrm{SD} = \sigma'$. Next we may extend $\bar{\mathrm{S}}$ onto $\hat{\mathfrak{A}}$ by defining $\bar{\mathrm{S}} = (\mathrm{S})'$ which is true on $\mathfrak{A}$ by the lemma. We may also extend the operator K^{-1} by letting $\mathrm{K}^{-1} = \mathrm{D}'$. While this is still an algebra homomorphism from $\hat{\mathfrak{A}}$ onto $\hat{\mathfrak{A}}$, it will not be an automorphism. Its kernel is the ideal generated by $\bar{x}_{-1}$. K does, therefore, not exist on $\hat{\mathfrak{A}}$ and a passage to $\bar{\mathrm{D}} = \mathrm{K}'$ is not possible. There is no consistent way to extend $\bar{\mathrm{D}}$ from $\mathfrak{A}$ to $\hat{\mathfrak{A}}$ so that for example $\bar{\mathrm{D}}\bar{\mathrm{S}} = \mathrm{I}$ holds, because $\bar{\mathrm{D}}\bar{\mathrm{S}}(\bar{x}_{-1}) = \bar{\mathrm{D}}(1) = 0 \neq \bar{x}_{-1}$. There is a natural way to extend $\bar{\mathrm{D}}$ onto $\hat{\mathfrak{A}}$, but without saving the just mentioned property and what's worse the mapping would not be surjective so that besides property (5) also property (2) would fail. Instead of violating (2) it is much better to violate (4). As a matter of fact, it is logical to ask the stronger question: Does there exist an inverse of $\bar{\mathrm{S}}$ on $\hat{\mathfrak{A}}$? It would have been futile to ask this question for $\bar{\mathrm{S}}$ restricted to $\mathfrak{A}$, because there we would have $\bar{\mathrm{D}}\bar{\mathrm{S}}\bar{\mathrm{S}}^{-1}(1) = 0$ so that $\bar{\mathrm{D}}\bar{\mathrm{S}}(1) = 1 = \bar{\mathrm{S}}(0) = 0$ if $\bar{\mathrm{S}}^{-1}$ would exist.

THEOREM 3. *The operator* $\bar{\mathrm{S}}$ *has an inverse* $\bar{\mathrm{S}}^{-1}$ *on* $\hat{\mathfrak{A}}$ *namely* $\bar{\mathrm{S}}^{-1} = \mathrm{I} + \mathrm{S} + \mathrm{S}^{(2)} + \cdots + \mathrm{S}^{(k)} + \cdots$ *and* $\bar{\mathrm{S}}^{-1}$ *satisfies conditions* (1)–(3) *for a semi-derivation. We have*: $\bar{\mathrm{S}}\bar{x}_i = \bar{x}_{i+1}$ *and* $\bar{\mathrm{S}}^{-1}\bar{x}_i = \bar{x}_{i-1}$ *for all integers i.*

Proof. (1) $\bar{\mathrm{S}}^{-1}$ is defined on $\hat{\mathfrak{A}}$ because if $c = \sum_{k=0}^{\infty}\gamma_k x_k$ then $\mathrm{S}^{(i)}(c) = \sum_{k=0}^{\infty}\gamma_k x_{k+i} \in (x_i)^*$ and $\bar{\mathrm{S}}^{-1}(c) = \sum_{k=0}^{\infty}\mathrm{S}^{(i)}(c)$ is convergent in $\hat{\mathfrak{A}}$.

(2) By our lemma we have: $\bar{\mathrm{S}} = \mathrm{I} - \mathrm{S}$ so that $\bar{\mathrm{S}}\bar{\mathrm{S}}^{-1} = (\mathrm{I} - \mathrm{S}) \cdot (\mathrm{I} + \mathrm{S} + \cdots + \mathrm{S}^{(k)} + \cdots) = \mathrm{I}$.

(3) Using the definition of $\bar{\mathrm{S}}^{-1}$ one computes without trouble

$$\bar{\mathrm{S}}^{-1}(x_i) = \sum_{k=i}^{\infty} x_k \, , \bar{\mathrm{S}}^{-1}(\bar{x}_i) = \bar{x}_{i-1} \qquad \text{and} \quad \bar{\mathrm{S}}^{-1}\left(\sum_{k=0}^{\infty}\gamma_k x_k\right) = \sum_{k=0}^{\infty}\gamma_k \bar{\mathrm{S}}^{-1}(x_k).$$

(4) It is clear that $\bar{\mathrm{S}}^{-1}$ is linear and surjective. We need to prove the product formula. We note first that using well-known properties of convergent series and the last formula in (3) above, that is sufficient to prove the

product formula for $x_i x_j$ for all $i, j > 0$. We should, therefore, prove that:

(A) $\bar{S}^{-1}(x_i x_j) = x_i \bar{S}^{-1}(x_j) + x_j \bar{S}^{-1}(x_i) - \bar{S}^{-1}(x_i)\bar{S}^{-1}(x_j)$.

One can do this straightforward by using induction, but it is simpler to use operators and their properties: From $SD = I - \sigma$ we obtain $(I - \bar{S})(I - K^{-1}) = I - \bar{S} - K^{-1} + \bar{S}K^{-1} = I - \sigma$ so that $\bar{S} + K^{-1} - \bar{S}K^{-1} = \sigma$. Multiplying by $\bar{S}^{-1}$ from the left, we get $I + \bar{S}^{-1}K^{-1} - K^{-1} = \bar{S}^{-1}\sigma = Q_{-1}\sigma$. Multiplying from the right by K (this is allowed as long as our operator equation is applied to elements in $\mathfrak{A}$), we obtain the following formula:

(B) $\bar{S}^{-1} = K - I + Q_{-1}\sigma K$ valid on $\mathfrak{A}$.

Computing the left side of (A) we get:

$$\bar{S}^{-1}(x_i x_j) = K(x_i x_j) - x_i x_j + Q_{-1}\sigma K(x_i x_j)$$

and computing the right side of (A) we get:

$$\begin{aligned}
&x_i \bar{S}^{-1}(x_j) + x_j \bar{S}^{-1}(x_i) - \bar{S}^{-1}(x_i)\bar{S}^{-1}(x_j) = x_i K(x_j) - x_i x_j + \\
&+ x_i Q_{-1}\sigma K(x_j) + x_j K(x_i) - x_i x_j + x_j Q_{-1}\sigma K(x_i) - \\
&- [K(x_i) - x_i + Q_{-1}\sigma K(x_i)] \cdot [K(x_j) - x_j + Q_{-1}\sigma K(x_j)] = \\
&= K(x_i)K(x_j) - x_i x_j + Q_{-1}\sigma K(x_j) \cdot K(x_i) + Q_{-1}\sigma K(x_i) \cdot \\
&\cdot K(x_j) - Q_{-1}\sigma K(x_i) \cdot Q_{-1}\sigma K(x_j) = K(x_i x_j) - \\
&- x_i x_j + Q_{-1}\sigma K(x_i x_j).
\end{aligned}$$

The last equality is obtained by using the fact that $\bar{x}_{-1}\bar{x}_{-1} = \bar{x}_{-1}$ and that K is an automorphism on $\mathfrak{A}$.

COROLLARY. *Changing all bared expressions to unbared ones and all unbared ones to bared ones and replacing* K *by* K^{-1} *and vice versa, we obtain—as above—a completion* $\bar{\hat{\mathfrak{A}}}$ *onto which the operators* $\bar{\sigma}$, $\bar{D}$, $\bar{S}$, S, K *may be extended and* S *has an inverse* S^{-1} *which is a semi-derivation satisfying conditions* (1)–(3). *The mapping* ${}^1J$ *from* $\mathfrak{A} \leftrightarrow \mathfrak{A}$ *may be extended to a mapping* ${}^1J$ *from* $\hat{\mathfrak{A}} \leftrightarrow \bar{\hat{\mathfrak{A}}}$ *and is an isomorphism sending unbared operators into bared ones and vice versa.*

Recall also the remark made at the beginning of Part II which guaranties an infinite number of other pairs of isomorphic completions $\{\hat{\mathfrak{A}}_m, \bar{\hat{\mathfrak{A}}}_m\}$. The isomorphism from $\hat{\mathfrak{A}}$ to $\hat{\mathfrak{A}}_m$ is an extension of K^m.

Next we would like to determine all **Z**-algebra automorphisms on $\hat{\mathfrak{A}}$. Let H be first a **Z**–algebra homomorphism on $\hat{\mathfrak{A}}$. It is clear that the recursion formulae (I) must hold. Let $H(x_j) = \sum_{k=0}^{\infty} \beta_{jk} x_k$ and let $a = \sum_{j=0}^{\infty} \alpha_j x_j$ be such that $H(a) = x_1$. For H to be an automorphism the system of equations $\sum_{j=k}^{\infty} \alpha_j \beta_{jk} = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$ must have a unique solution in the α_j's. Since the partial system with $k \geq 2$ is homogeneous, we must have $\alpha_j = 0$ for all $j \geq 2$. It follows that $\alpha_1 \beta_{11} = 1$ which means that $\alpha_1 = \beta_{11} = 1$ or -1 so that $H(x_1) = \beta_{10} \pm x_1$ and H must be an extension of an automorphism $A \in \mathbf{A}$. Assume $A = K^{-m}$ i.e., $\beta_{10} = -m$ then $\beta_{ik} = \binom{-m+i-1-k}{i-k}$. Now $\bar{x}_{-1} = \sum_{i=0}^{\infty} x_i$ must have an image $\sum_{k=0}^{\infty} \gamma_k x_k$ with $\gamma_k = \sum_{k=0}^{\infty} \binom{-m+i-1-k}{i-k}$. This sum exists if and only if $m \geq 0$. Similarly in the case where $A = JK^{-m}$ the coefficient of x_k is $\sum_{i=0}^{\infty} (-1)^k \binom{-m+i-1}{i-k}$ which never exists for all k. The only automorphisms $A \in \mathbf{A}$ which may be extended to a $\mathbf{Z}$-algebra homomorphism onto $\hat{\mathfrak{A}}$ are $A = K^{-m}$ for $m \geq 0$. What is the kernel of K^{-m}? Let $b = \sum_{i=0}^{\infty} \beta_i x_i$ such that $K^{-m} b = 0$. This leads to the homogeneous linear system of equations: $\sum_{i=k}^{\infty} \binom{-m+i-1-k}{i-k} \beta_i = 0$. The $\beta_0, \cdots, \beta_{m-1}$ may be freely chosen and the β_i with $i \geq m-1$ are then determined so that Ker K^{-m} is a $(m-1)$-dimensional subspace of $\hat{\mathfrak{A}}$. We assert that $\{\bar{x}_{-1}, \bar{x}_{-2}, \cdots \cdots, \bar{x}_{-m+1}\}$ generates this subspace. Note first that from $\bar{S}\bar{x}_{-i} = \bar{x}_{-i+1}$ and $\bar{S} = I - S$ we get $\bar{x}_{-i} = \bar{x}_{-i+1} + S\bar{x}_{-i}$. Now $K^{-1} = I - D$ and therefore $K^{-1}(\bar{x}_{-i}) = \bar{x}_{-i+1} + S\bar{x}_{-i} = K^{-1}(\bar{x}_{-i+1}) + S\bar{x}_{-i} - \bar{x}_{-i} = K^{-1}(\bar{x}_{-i+1}) - \bar{x}_{-i+1}$ and from this we get by repetition: $K^{-1}(\bar{x}_{-i}) = -(\bar{x}_{-1} + \bar{x}_{-2} + \cdots + \bar{x}_{-m+1})$. Therefore $K^{-m}(\bar{x}_{-i}) = 0$ for $i \leq m-1$ and it follows that $\bar{x}_{-i} \in$ Ker K^{-m} for $i \leq m-1$. Since $\bar{x}_{-1}, \bar{x}_{-2}, \cdots, \bar{x}_{-m+1}$ are clearly independent elements of $\hat{\mathfrak{A}}$ our assertion is proved.

THEOREM 4. *The only* $\mathbf{Z}$*-algebra automorphism on* $\hat{\mathfrak{A}}$ *is the identity. The only automorphisms on* $\mathfrak{A}$ *which may be extended as homomorphisms onto* $\hat{\mathfrak{A}}$ *are the* K^{-m} *with* $m \geq 0$ *and* Ker $K^{-m} = (\bar{x}_{-1}, \bar{x}_{-2}, \cdots, \bar{x}_{-m+1})$.

We are now in a position to construct other semiderivations on $\hat{\mathfrak{A}}$. Let $D_m = I - K^{-m}$. (Warning: This is by no means an extension of $D_m = K^m D K^{-m} = D$ which was defined on $\mathfrak{A}$, but cannot be extended to $\hat{\mathfrak{A}}$. The simplicity of the notation warrants its use and no confusion should result).

ASSERTION. D_m *is a semi-derivation on* $\hat{\mathfrak{A}}$.

Proof. Properties (1), (3), and (4) clearly hold. We must prove property (2); namely, that D_m is *surjective*. Let $a = \sum_{i=0}^{\infty} \alpha_i x_i \in \hat{\mathfrak{A}}$. We must show that the differential equation $D_m y = a$ has a solution. For that it will be sufficient to prove that (1) $D_m y = x_i$ has a solution y_i for all $i = 0, 1, 2, \cdots$ and (2) that $\{y_i\} \to 0$, because $y = \sum_{i=0}^{\infty} \alpha_i y_i$ exists then and $D_m y = a$.

(1) Let $C_m = \sum_{k=1}^{\infty} (-1)^{k+1} \binom{m}{k} D^{k-1}$. Since $D_m = I - K^{-m} = I - (I - D)^m$ we have: $D_m = C_m D = DC_m$. We show that: $C_m y = 0$ has an

$(m-1)$-dimensional solution space. Put $y = \sum_{i=0}^{\infty} \beta_i x_i$. We must have: $\sum_{k=1}^{\infty} (-1)^{k+1} \binom{m}{k} \beta_{r+k-1} = 0$ for every $r = 0, 1, 2, \cdots$ To solve this system of equations, we may choose $\beta_0, \beta_1, \cdots, \beta_{m-2}$ arbitrarily and the β_i for $i \geq m-1$ are then uniquely determined. Let ρ_m be that solution for which $\beta_0 = \beta_1 = \cdots = \beta_{m-3} = 0$ and $\beta_{m-2} = 1$. Put $y_i = (-1)^{m+1} S^{(i+2)} \rho_m$. *Assertion:* $D_m y_i = x_i$. *Proof:* $D_m S\rho_m = C_m \rho_m = 0$. Now $D_m S^2 \rho_m = C_m S\rho_m$ and $S(D_m S\rho_m) = SD(C_m S\rho_m) = C_m S\rho_m - \sigma C_m S\rho_m = 0$ but $\sigma C_m S\rho_m = (-1)^{m+1}$ since $S\rho_m = x_{m-1} + \cdots$ Therefore $D_m y_0 = 1$. Now—using induction—assume: $D_m y_{i-1} = x_{i-1}$. $D_m S^{i+2} \rho_m = C_m S^{i+1} \rho_m$ and $S(D_m S^{i+1} \rho_m) = C_m S^{i+1} \rho_m - \sigma C_m S^{i+1} \rho_m = (-1)^{m+1} x_i$ but $\sigma C_m S^{i+1} \rho_m = 0$ for $i \geq 1$ so that $D_m y_i = x_i$.

(2) Since $y_i = \pm x_i + \cdots$ it follows at once that $\{y_i\} \to 0$ in our topology.

As mentioned under section (1) in the proof above, the complete solution of the differential equation $C_m y = 0$ is a $(m-1)$-dimensional linear subspace of $\hat{\mathfrak{A}}$. It is generated by the independent solutions ρ_{mj} which are obtained by putting $\beta_i = 0$ for $0 \leq i \leq m-2$, $i \neq j$ and $\beta_j = 1$ in the general solution. It is then clear that $D_m y = 0$ has the m-dimensional solution-space $\mathbf{Z}_m = \operatorname{Ker} D_m = (1, S\rho_{m0}, S\rho_{m1}, \cdots, S\rho_{m\,m-2})$. Since $D_m(ab) = aD_m b + bD_m a - D_m aD_m b = 0$ for $a, b \in \mathbf{Z}_m$, it follows that $\mathbf{Z}_m$ is actually a $\mathbf{Z}$-subalgebra of $\hat{\mathfrak{A}}$. The elements of $\mathbf{Z}_m$ are the *constants* with respect to the semi-derivation D_m and we may consider $\hat{\mathfrak{A}}$ as a $\mathbf{Z}_m$-algebra.

Suppose that H is a $\mathbf{Z}_m$-algebra homomorphism from $\hat{\mathfrak{A}}$ onto $\mathbf{Z}_m$. We may then define a semi-integration S_H with respect to D_m by letting: $S_H(a) = a' - H(a')$ for $a \in \hat{\mathfrak{A}}$, where a' is any element of $\hat{\mathfrak{A}}$ such that $D_m a' = a$. $S_H(a)$ is well defined, because if a'' is another element such that $D_m a'' = a$, it follows that $D_m(a'' - a') = 0$ or $a'' - a' \in \mathbf{Z}_m$ so that $H(a'' - a') = H(a'') - H(a') = a'' - a'$ or $a'' - H(a'') = a' - H(a')$. Clearly we have: $D_m S_H = I$ and $S_H D_m = I - H$ and for different H the S_H differ at most by a constant, i.e. an element of $\mathbf{Z}_m$. Imitating further, we may define: $\overline{S}_H = (S_H)'$ and finally $\overline{S}_H^{-1} = I + S_H + S_H^2 +, \cdots$, the *inverse* of $\overline{S}_H$, which is—proof the same as for $\overline{S}^{-1}$—*a* semi-derivation without property (4) on $\hat{\mathfrak{A}}$. We state all this in the following theorem.

THEOREM 5. *To each natural number m there exists a semiderivation* $D_m = (K^{-m})'$ *on* $\hat{\mathfrak{A}}$. *If* H *is a* $\mathbf{Z}_m$*-algebra homomorphism from* $\hat{\mathfrak{A}}$ *onto* $\mathbf{Z}_m$, *where* $\mathbf{Z}_m = \operatorname{Ker} D_m$ (*the sub* $\mathbf{Z}$*-algebra of constants for* D_m), *then there exists a semi-integration* $\overline{S}_H$ *for* D_m *with the properties*: $D_m S_H = I$, $S_H D_m = H'$ *and two different* S_H*'s differ at most by a constant.* $\overline{S}_H = (S_H)'$ *has an inverse* $\overline{S}_H^{-1} = \sum_{i=0}^{\infty} S_H^i$ *which is a semi-derivation satisfying properties* (1)–(3).

COROLLARY. *Let* $z_{Hi} = S_H^i(1)$ *then* $\{z_{Hi}\}$ *forms a complete basis for the* $\mathbf{Z}_m$*-algebra* $\hat{\mathfrak{A}}$.

Proof. $x_1 = S_H D_m x_1 + H(x_1) = S_H C_m(1) + H(x_1) = S_H(m) + H(x_1)$. Therefore $x_1 = mz_{H1} + H(z_1)$

$$x_2 = S_H D_m x_2 + H(x_2) = S_H C_m(x_1) + H(x_2) = S_H\left[mx_1 - \binom{m}{2}\right] + $$
$$+ H(x_2) = S_H\left[m^2 z_{H1} + mH(x_1) - \binom{m}{2}\right] + H(x_2) = m^2 z_{H2} +$$
$$+ \left[mH(x_1) - \binom{m}{2}\right] z_{H1} + H(x_2)$$

and in general wa may express similarly

$$x_i = \beta_{ii} z_{Hi} + \beta_{ii-1} z_{Hi-1} + \cdots + \beta_{i0}$$

where $\beta_{ik} \in \mathbf{Z}_m$. β_{ik} contains terms of the form $P(m)H(x_j)$ where $P(X)$ is a polynomial over $\mathbf{Z}$ and if such a term occurs in β_{i-1k} then $P(m)H(x_{j+1})$ will occur in β_{ik}. From this it follows that if $a = \sum_{i=0}^{\infty} \alpha_i x_i \in \hat{\mathfrak{A}}$ then $a = \sum_{k=0}^{\infty} \gamma_k z_{Hk}$ with $\gamma_k = Q(m; H(b_i)) \in \mathbf{Z}_m$ where $Q(W; Y_i)$ is a polynomial over $\mathbf{Z}$ and linear in the Y_i.

The problem arises to determine all possible $\mathbf{Z}_m$-algebra homomorphisms from $\hat{\mathfrak{A}}$ onto $\mathbf{Z}_m$. We will solve this problem here for the case $m = 2$. Let us first learn how to compute in $\mathbf{Z}_2$. We need for this purpose the element ρ_2. According to its definition, we find its series expansion: $\rho_2 = \sum_{i=0}^{\infty} 2^i x_i$. Let $e = S\rho_2$ then we know that $\mathbf{Z}_2 = \{\alpha_0 + \alpha_1 e \mid \alpha_0, \alpha_1 \in \mathbf{Z}\}$ is the $\mathbf{Z}$-vector space representation of $\mathbf{Z}_2$. Note that $e^2 = -e$. This relation determines the multiplication: $(\alpha_0 + \alpha_1 e)(\beta_0 + \beta_1 e) = \alpha_0\beta_0 + (\alpha_0\beta_1 + \beta_0\alpha_1 - \alpha_1\beta_1)e$. It is interesting to note that $\rho_2 = 1 + 2e$ is an element of $\mathbf{Z}_2$ and that $(\rho_2)^2 = 1$. Now let H be a $\mathbf{Z}_2$-algebra homomorphism from $\hat{\mathfrak{A}}$ onto $\mathbf{Z}_2$. It will be uniquely determined by giving $H(x_1) = \beta_0 + \beta_1 e$ but we cannot choose the β_0 and β_1 arbitrarily in $\mathbf{Z}$, because we must also have $H(e) = e$. To see what restrictions this imposes we use formula (I) and induction to find:

$$H(x_i) = \binom{\beta_0 + i - 1}{i} + \left[\sum_{k=1}^{\beta_1} (-1)^{k+1} \binom{\beta_0 + i - 1 - k}{i - k} \binom{\beta_1}{k}\right] e.$$

This in turn gives

$$H(e) = \sum_{i=1}^{\infty} 2^{i-1} H(x_i) = \sum_{i=1}^{\infty} \binom{\beta_0 + i - 1}{i} 2^{i-1} +$$
$$+ \left[\sum_{k=1}^{\infty} (-1)^{k+1} \binom{\beta_1}{k} \left(\sum_{i=1}^{\infty} \binom{\beta_0 + i - 1 - k}{i - k} 2^{i-1}\right)\right] e = 0 + 1 . e.$$

For the constant term to be zero, we are forced to choose β_0 either to be zero or a negative even integer; this turns the bracket above into

$$\sum_{k=1}^{\infty} (-1)^{k+1} \binom{\beta_1}{k} 2^{k-1} = 1/2 \, [1-(-1)^{\beta_1}]$$

which is $= 1$ if and only if β_1 is a positive, odd integer.

THEOREM 6. *The only possible* $\mathbf{Z}_2$*–algebra homomorphisms from* $\hat{\mathfrak{A}}$ *onto* $\mathbf{Z}_2$ *are those which map* x_1 *into the elements* $\beta_0 + \beta_1 e$ *where* β_0 *is an even, non-positive integer and* β_1 *is an odd, positive integer.*

LITERATURE

[1] E. G. KUNDERT (1966) – *Structure Theory in s–d–Rings*. Nota I, « Acc. Naz. Lincei », ser. VIII *41*.

[2] T. GIEBUTOWSKI (1971) – *Ph. D. Thesis*, Univ. of Mass. (will be published in the « Journal of Algebra »).