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An oscillation criterion for a fourth order integrally superlinear differential equation

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Equazioni differenziali ordinarie. — *An oscillation criterion for a fourth order integrally superlinear differential equation.* Nota di
DAVID LOWELL LOVELADY, presentata (*) dal Socio C. SANSONE.

RIASSUNTO. — L'Autore dimostra che sotto opportune condizioni tutte le soluzioni dell'equazione non lineare del quarto ordine $(r(t)\, u''(t))'' + q(t) f(u(t)) = 0$ sono oscillatorie.

I. INTRODUCTION AND RESULTS

Several Authors (see, for example, F. V. Atkinson [1], J. W. Macki and J. S. W. Wong [3], and G. H. Ryder and D. V. V. Wend [4]) have obtained oscillation theorems, for equations such as

$$u^{(n)}(t) + q(t) f(u(t)) = 0\,,$$

based on the idea of hypothesizing for q enough to ensure that every *bounded* solution of

$$u^{(n)}(t) + q(t)\, u(t) = 0$$

is oscillatory and hypothesizing that f is *integrally superlinear* in the sense that if $\alpha > 0$ then

$$\int_{\alpha}^{\infty} f(x)^{-1}\, dx < \infty \tag{1}$$

and

$$\int_{-\infty}^{-\alpha} f(x)^{-1}\, dx > -\infty\,. \tag{2}$$

The extension of this type of result to the second order equation

$$(r(t)\, u'(t))' + q(t) f(u(t)) = 0$$

is easily accomplished through a Liouville transformation (see Wong [5, § 6]). In the present note we shall show that the same type of result holds for the fourth order equation

$$(r(t)\, u''(t))'' + q(t) f(u(t)) = 0\,. \tag{3}$$

(*) Nella seduta del 12 aprile 1975.

In particular, we shall prove the following theorem.

THEOREM. *Let r be a continuous function from* $[0, \infty)$ *to* $(0, \infty)$ *and let q be a continuous function from* $[0, \infty)$ *to* $[0, \infty)$. *Suppose that*

$$\int_0^\infty r(s)^{-1} \, ds = \infty , \tag{4}$$

and

$$\int_0^\infty \left(\int_0^t (t - s) \, sr(s)^{-1} \, ds \right) q(t) \, dt = \infty . \tag{5}$$

Let f be a continuous nondecreasing function from $(-\infty, \infty)$ *to* $(-\infty, \infty)$ *such that* $xf(x) > 0$ *if* $x \neq 0$ *and such that* (1) *and* (2) *are true if* $\alpha > 0$. *Then every solution of* (3) *is oscillatory.*

Our concept of solution here us as follows: If z is in $[0, \infty)$, u is a continuous function from $[z, \infty)$ to $(-\infty, \infty)$, u is twice differentiable on (z, ∞), ru'' is twice differentiable on (z, ∞), and (3) is true whenever $t > z$, then u is said to be a solution of (3). Note that it follows from [2, Theorem 1] that (4) and (5) ensure the oscillation of every bounded solution of

$$(ru'')'' + qu = 0 .$$

Also, it follows from [2, Theorem 3] that, with the hypotheses of our present theorem, every bounded solution of (3) is oscillatory.

II. PROOFS

Let u be a nonoscillatory solution of (3). If u is eventually negative then $-u$ is a solution of

$$(r(t) \, v''(t))'' + q(t) \, f^*(v(t)) = 0 ,$$

where f^* is given by $f^*(x) = -f(-x)$. But f^* satisfies all the hypotheses required of f, so we may assume that u is eventually positive. Find a in $[z, \infty)$ such that u is positive on $[a, \infty)$. On $[a, \infty)$, let $v_1 = u$, $v_2 = u'$, $v_3 = ru''$, and $v_4 = (ru'')'$. Now

$$\begin{aligned} v_1' &= v_2 \\ v_2' &= v_3/r \\ v_3' &= v_4 \\ v_4' &= -qf(v_1) \end{aligned} \tag{6}$$

on $[a, \infty)$. Thus v_4 is nonincreasing on $[a, \infty)$. If v_4 is ever negative, then (6) and (4) say that v_1 is eventually negative, a contradiction. Thus $v_4(t) \geq 0$ whenever $t \geq a$, and $\lim_{t \to \infty} v_4(t) = v_4(\infty)$ exists, $v_4(\infty) \geq 0$. Also, $v_4(t) > 0$

if $t > a$. For if $b > a$ and $v_4(b) = 0$, then $v_4(t) = 0$ whenever $t \geq b$. Thus $v_4'(t) = 0$ and $q(t) = 0$ whenever $t > b$. But this contradicts (5), so $v_4(t) > 0$ whenever $t > a$. Thus v_3 is increasing on $[a, \infty)$. Now either $v_3 < 0$ on $[a, \infty)$, or there is $b > a$ such that $v_3 > 0$ on $[b, \infty)$. First suppose $v_3 < 0$ on $[a, \infty)$. Now $v_4(\infty) = 0$, since $v_4(\infty) > 0$ would yield a contradiction. Also, v_2 is decreasing on $[a, \infty)$, and v_2 can never be negative just as v_4 above could never be negative. Thus $v_2(\infty) > 0$. Since $v_2(\infty)$ exists, (4) and (6) say that $v_3(\infty) = 0$. Since $v_4(\infty) = 0$, if $\tau \geq t \geq a$ then

$$v_4(\tau) - v_4(t) = -\int_t^\tau q(s) f(u(s))\, ds ,$$

so

$$v_4(t) = \int_t^\infty q(s) f(u(s))\, ds .$$

Similarly,

$$-v_3(t) = \int_t^\infty v_4(s)\, ds ,$$

$$v_3(t) = -\int_t^\infty \left(\int_s^\infty q(\xi) f(u(\xi))\, d\xi \right) ds$$

if $t \geq a$. If $\tau \geq t \geq a$ then

$$v_2(\tau) - v_2(t) = \int_t^\tau (v_3(s)/r(s))\, ds ,$$

so, since $v_2(\infty) \geq 0$,

$$v_2(t) \geq -\int_t^\infty (v_3(s)/r(s))\, ds$$

$$= \int_t^\infty \left(r(s)^{-1} \int_s^\infty \left(\int_\xi^\infty q(\sigma) f(u(\sigma))\, d\sigma \right) d\xi \right) ds .$$

Thus, if $t \geq a$,

$$u(t) = u(a) + \int_a^t v_2(s)\, ds$$

$$\geq \int_a^t v_2(s)\, ds$$

$$\geq \int_a^t \left(\int_s^\infty \left(r(\xi)^{-1} \int_\xi^\infty \left(\int_\sigma^\infty q(\tau) f(u(\tau))\, d\tau \right) d\sigma \right) d\xi \right) ds$$

$$= (t-a) \int_t^\infty \left(r(s)^{-1} \int_s^\infty \left(\int_\xi^\infty q(\sigma) f(u(\sigma))\, d\sigma \right) d\xi \right) ds$$

$$+ \int_a^t \left((s-a)\, r(s)^{-1} \int_s^\infty \left(\int_\xi^\infty q(\sigma) f(u(\sigma))\, d\sigma \right) d\xi \right) ds$$

$$\geq \int_a^t \left((s-a)\, r(s)^{-1} \int_s^\infty \left(\int_\xi^\infty q(\sigma) f(u(\sigma))\, d\sigma \right) d\xi \right) ds$$

$$= \left(\int_a^t (s-a)\, r(s)^{-1}\, ds \right) \left(\int_t^\infty \left(\int_s^\infty q(\xi) f(u(\xi))\, d\xi \right) ds \right)$$

$$+ \int_a^t \left(\int_a^s (\xi - a)\, r(\xi)^{-1}\, d\xi \right) \left(\int_s^\infty q(\xi) f(u(\xi))\, d\xi \right) ds$$

$$\geq \int_a^t \left(\int_a^s (\xi - a)\, r(\xi)^{-1}\, d\xi \right) \left(\int_s^\infty q(\xi) f(u(\xi))\, d\xi \right) ds$$

$$= \left(\int_a^t (t-s)(s-a)\, r(s)^{-1}\, ds \right) \left(\int_t^\infty q(s) f(u(s))\, ds \right)$$

$$+ \int_a^t \left(\int_a^s (s-\xi)(\xi-a)\, r(\xi)^{-1}\, d\xi \right) q(s) f(u(s))\, ds$$

$$\geq \int_a^t \left(\int_a^s (s-\xi)(\xi-a)\, r(\xi)^{-1}\, d\xi \right) q(s) f(u(s))\, ds\,.$$

Now L'Hopital's Rule says that

$$\lim_{s\to\infty} \frac{\int_a^s (s-\xi)(\xi-a)\, r(\xi)^{-1}\, d\xi}{\int_0^s (s-\xi)\, \xi r(\xi)^{-1}\, d\xi} = 1\,,$$

so (5) ensures

(7) $$\int_a^\infty \left(\int_a^t (t-\xi)(\xi-a)\, r(\xi)^{-1}\, d\xi \right) q(t)\, dt = \infty\,.$$

Let φ be given on $[a, \infty)$ by

$$\varphi(t) = q(t) \int_a^t (t - s)(s - a) r(s)^{-1} \, ds .$$

Now the above computations show that

$$u(t) \geq \int_a^t \varphi(s) f(u(s)) \, ds$$

whenever $t \geq a$. Let w be given on $[a, \infty)$ by

$$w(t) = \int_a^t \varphi(s) f(u(s)) \, ds .$$

Now, if $t > a$,

$$w'(t) = \varphi(t) f(u(t)) \geq \varphi(t) f(w(t)) ,$$

so, if $t \geq b > a$,

$$\int_b^t w'(s) f(w(s)) \, ds \geq \int_b^t \varphi(s) \, ds ,$$

$$\int_{w(b)}^{\infty} f(x)^{-1} \, dx \geq \int_{w(b)}^{w(t)} f(x)^{-1} \, dx \geq \int_b^t \varphi(s) \, ds .$$

But (1) and (7) say that this is impossible, so we have a contradiction. Thus, in the case "$v_3 < 0$ on $[a, \infty)$", the proof is complete.

Suppose that there is $b > a$ such that $v_3 > 0$ on $[b, \infty)$. Now v_2 is increasing on $[b, \infty)$. Since v_3 is increasing,

$$v_2(t) = v_2(b) + \int_b^t (v_3(s)/r(s)) \, ds$$

$$\geq v_2(b) + v_3(b) \int_b^t r(s)^{-1} \, ds$$

whenever $t \geq b$. Thus (4) says there is $c \geq b$ such that $v_2 > 0$ on $[c, \infty)$. Now, if $t \geq c$,

$$u(t) = u(c) + \int_c^t v_2(s) \, ds$$

$$\geq \int_c^t v_2(s) \, ds$$

$$= \int_c^t \left(v_2(c) + \int_c^s (v_3(\xi)/r(\xi))\, d\xi \right) ds$$

$$\geq \int_c^t \left(\int_c^s (v_3(\xi)/r(\xi))\, d\xi \right) ds$$

$$= \int_c^t ((t-s)\, v_3(s)/r(s))\, ds$$

$$= \int_c^t (t-s)\, r(s)^{-1} \left(v_3(c) + \int_c^s v_4(\xi)\, d\xi \right) ds$$

$$\geq \int_c^t (t-s)\, r(s)^{-1} \left(\int_c^s v_4(\xi)\, d\xi \right) ds$$

$$\geq \int_c^t (t-s)\, r(s)^{-1} \left(\int_c^s \left(\int_\xi^\infty q(\sigma) f(u(\sigma))\, d\sigma \right) d\xi \right) ds$$

$$\geq \int_c^t (t-s)\, r(s)^{-1} (s-c) \left(\int_s^\infty q(\xi) f(u(\xi))\, d\xi \right) ds$$

$$= \left(\int_c^t (t-s)(s-c)\, r(s)^{-1}\, ds \right) \left(\int_t^\infty q(s) f(u(s))\, ds \right)$$

$$+ \int_c^t \left(\int_c^s (s-\xi)(\xi-c)\, r(\xi)^{-1}\, d\xi \right) q(s) f(u(s))\, ds$$

$$\geq \int_c^t \left(\int_c^s (s-\xi)(\xi-c)\, r(\xi)^{-1}\, d\xi \right) q(s) f(u(s))\, ds\,.$$

Now an argument virtually identical to that used in the first case gives a contradiction, and the proof is complete.

References

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[5] J. S. W. Wong (1968) – *On second order nonlinear oscillation*, « Funkcialaj Ekvacioj », *11*, 207–234.