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A theorem of existence and uniqueness for an integral equation in topological spaces

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Equazioni funzionali. — *A theorem of existence and uniqueness for an integral equation in topological spaces.* Nota di SENDER SOLOMON, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore perviene ad un teorema di esistenza e di unicità più generale di quelli finora noti.

We have studied in [5], [6] a generalization of the differential equation for set functions of A. Haimovici [2], [3]; here we give a new theorem of existence and uniqueness.

Let be X a topological space, μ a positive measure such that every compact is measurable, Γ the class of all relatively compact subsets with finite measure; let be P and Q applications from X to Γ satisfying the following conditions:

$$\lim \mu (Px \,\Delta y) = 0 \qquad \text{when} \quad y \to x , \tag{1}$$

$$Px \subset Qx , \tag{2}$$

$$Qx \subset Qy \qquad \text{for} \quad x \in Qy ; \tag{3}$$

let be E a Banach space and f a continuous application from $X \times X \times E$ to E with the property

$$|f(x, y, z) - f(x, y, t) \leqq a(x, y) \,|z - t| . \tag{4}$$

THEOREM. *If* X *is Hausdorff, each* Qx *is compact and the equation*

$$w(x) = 1 + \int_{Qx} a(x, y)\, w(y)\, dy \tag{5}$$

has a positive solution $w : X \to R$ *such that each restriction* $w|_{Qz}$ *is continuous, then the equation*

$$u(x) = u_0(x) + \int_{Px} f(x, y, u(y))\, dy \tag{6}$$

with $u_0 : X \to E$ *continuous, has a unique solution* $u : X \to E$ *such that each restriction* $u|_{Qz}$ *is continuous.*

Proof. We first suppose that X is compact and w is continuous. We consider the operator

$$Au(x) = u_0(x) + \int_{Px} f(x, y, u(y))\, dy$$

(*) Nella seduta dell'11 giugno 1975.

defined on C, the space of all continuous functions $u : X \to E$ normed by

$$|u| = \sup \frac{|u(x)|}{w(x)}$$

C is a Banach space because $|u| \leq \sup |u(x)| \leq |u| \sup |w(x)|$. We show that $A(C) \subset C$. Let $u \in C$ and $x_0 \in X$. Then

$$|Au(x) - Au(x_0)| \leqq |u_0(x) - u_0(x_0)| +$$

$$+ \left| \int_{Px} f(x, y, u(y))\, dy - \int_{Px_0} f(x, y, u(y))\, dy \right| +$$

$$+ \left| \int_{Px_0} f(x, y, u(y))\, dy - \int_{Px_0} f(x_0, y, u(y))\, dy \right| \leqq |u_0(x) - u_0(x_0)| +$$

$$+ \int_{Px \Delta Px_0} |f(x, y, u(y))|\, dy + \int_{Px_0} |f(x, y, u(y)) - f(x, y, u(y))|\, dy .$$

Since $(x, y) \to f(x, y, u(y))$ is an uniformly continuously application from the compact $X \times X$ to E we can find, for given $\varepsilon > 0$, a neighbourhood V of x_0 such that $|f(x, y, u(y)) - f(x_0, y, u(y))| < \varepsilon$ for $x \in V$ and $y \in X$; besides $M = \sup |f(x, y, y(y))|$ is finite. So we have

$$|Au(x) - Au(x_0)| \leqq |u_0(x) - u_0(x_0)| + M\mu(Px \Delta Px_0) + \varepsilon\mu(Px_0) \qquad \text{for} \quad x \in V$$

which implies that Au is coutinuous.

A is a contraction: if $u, v \in C$ then

$$|Au(x) - Av(x)| \leqq \int_{Px} a(x, y) |u(y) - u(v)| dy \leqq |u - v| \int_{Px} a(x, y) w(y)\, dy \leqq$$

$$\leqq |u - v| \int_{Qx} a(x, y) w(y)\, dy = |u - v| (w(x) - 1)$$

and

$$|Au - Av| \leq |u - v| \sup \frac{w(x) - 1}{w(x)} .$$

It follows that (6) has a unique continuous solution if X is compact and w is continuous. In the general case, by (2) and (3), we can restrict (6) to Qz and we find a unique continuous solution $u_z : Qz \to E$. The restriction of (6) to $Qz \cap Qt$ has also a unique continuous solution. Hence $u_z(x) = u_t(x)$ on

$Qz \cap Qt$. We see that (6) has a unique solution on $\cup Qx \supset \cup Px$ with $u|_{Qz}$ continuous and this solution can be extended uniquely to the whole X by (6) itself. The proof is complete.

Remark 1. It is obvious from the proof that we can replace Q by P in (5).

Remark 2. If the interiors of all Qz form a covering of X then the solution is continuous on X. Moreover, then we can give up the conditions that X is Hausdorff and each Qx is compact. This is based upon two assertions:

LEMMA 1. *Let* Y *be a complete Hausdorff locally convex space defined by the family of seminorms* $\{\,|\;|_\alpha\}$ *and let* A *be an operator on* Y *which has the contraction property: for each* α *there exists* $q_\alpha \in (0, 1)$ *such that* $|Ax - Ay|_\alpha \leqq q_\alpha |x - y|_\alpha$. *Then* A *has one and only one fixed point (A more general result was obtained by G. Marinescu in* [4]).

LEMMA 2. *Let* X *be a topological space*, Σ *a familly of subsets of* X, *and* Y *the space of all functions* $u : X \to E$ *such that* $u|_S$ *is continuous for each* $S \in \Sigma$. *Then* Y *equipped with the* Σ*-convergence topology is complete* (see [1], Cor. 3, of Th. 2).

For the proof of the last remark we take $\Sigma = \{Qz\,;\, z \in X\}$ and Y the space of all continuous functions equipped with the family of seminorms

$$|u|_z = \sup_{x \in Qz} \frac{|u(x)|}{w(x)} \,.$$

An example. Let $X = R^n$ with the Lebesgue measure. For every $x = (x_1, \cdots, x_n) \in R^n$ we define

$$Qx = \mathop{\mathbf{X}}_{i=1}^{n} [\alpha(x_i), \beta(x_i)] \tag{7}$$

where $\alpha(t) = t$ for $t < 0$, $\alpha(t) = 0$ for $t \geqq 0$ and $\beta(t) = t - \alpha(t)$. The application Q verifies all the conditions required by the theorem. For exactness we shall verify that (5) has a continuous positive solution. It is sufficient to show this for its restriction to each Qz. We consider now the operator

$$Aw(x) = 1 + \int_{Qx} a(x, y)\, w(y)\, dy$$

(with a continuous, for instance) acting on the space Y of all continuous real functions defined on Qz with the norm

$$|w| = \sup_{x \in Qz} |w(x)|\, e^{-L(|x_1| + \cdots + |x_n|)}$$

where $L^n \geqq \sup \{a(x, y); x, y \in Qz\} = M$, and we prove that A is a contraction Let it be, for instance, $z_1 < 0, z_2 > 0, \cdots, z_n > 0$; then for $w, \bar{w} \in Y$,

$$|Aw(x) - A\bar{w}(x)| \leqq \int_{Qx} a(x, y) |w(y) - \bar{w}(y)| dy \leqq$$

$$\leq |w - \bar{w}| M \int_{Qx} e^{L(|y_1| + \cdots + |y_n|)} dy \leqq$$

$$= |w - \bar{w}| M \int_{x_1}^{0} e^{-Ls} ds \int_{0}^{x_2} e^{Ls} ds \cdots \int_{0}^{x_n} e^{Ls} ds =$$

$$= |w - w| \frac{M}{L^n} (e^{-Lx_1} - 1)(e^{Lx_2} - 1) \cdots (e^{Lx_n} - 1) \leqq$$

$$\leqq |w - \bar{w}| \frac{M}{L^n} e^{L(|x_1| + \cdots + |x_n|)}$$

hence $|Aw - A\bar{w}| \leqq \frac{M}{L^n} |w - \bar{w}|$ q.e.d.

Remark 3. If we take $X = R^n_+ = \{x = (x_1, \cdots, x_n); x_1 \geqq 0, \cdots, x_n \geqq 0\}$ an application P satisfying (1), (2) and Q given by (7) we obtain then one of the equations studied by W. Walter (see e.g. [7]).

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