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Absolute Riesz summability

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Analisi numerica. — *Absolute Riesz summability* (*). Nota (**) di PREM CHANDRA, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Si generalizzano due teoremi dovuti all'Autore sulle serie di Fourier.

I. DEFINITIONS AND NOTATIONS

Let $L(w)$ be a continuos, differentiable and monotonic increasing function of w, and let it tend to infinity with w. Suppose that Σa_n be a given infinite series [1] then

$$\Sigma a_n \in | R , L(w) , r | \qquad (r > 0)$$

if (see Mohanty [4])

$$\int_h^\infty \frac{L'(w)}{(L(w))^{1+r}} \left| \sum_{n<w} (L(w) - L(n))^{r-1} L(n) a_n \right| dw$$

is convergent, where h is a positive number (Obrechkoff [5, 6]) and $L'(w) = \frac{d}{dw} L(w)$.

We define the summability $| R , L(w) , 0 |$ equivalent to the absolute convergence.

Let f be 2π-periodic function and L-integrable over $(-\pi , \pi)$. We assume, without any loss of generality, that the Fourier series of f, at a point $t = x$, is

$$\Sigma (a_n \cos nx + b_n \sin nx) \equiv \Sigma A_n(x) . \tag{1.1}$$

The series conjugate to (1.1) is

$$\Sigma (b_n \cos nx - a_n \sin nx) \equiv \Sigma B_n(x) . \tag{1.2}$$

(*) This paper is dedicated to the memory of my mother who left the physical world on 30 may, 1975.

(**) Pervenuta all'Accademia il 25 luglio 1975.

(1) Summations are over $1, 2, \cdots, \infty$ when there is no indication to the contrary.

Throughout the paper we write

(1.3) $\varphi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$.

(1.4) $\psi(t) = \frac{1}{2}\{f(x+t)-f(x-t)\}$.

(1.5) $e(w) = \exp\{w/(\log w)^c\} \qquad (c > 0, \omega \geq A > 1)$.

(1.6) $h(n) = (\log(n+1))^d \qquad (d \geq 0)$.

(1.7) $e^{(1)}(w) = \frac{d}{dw}(e(w))$.

(1.8) $e^a(w) = (e(w))^a \qquad$ (for finite a).

(1.9) $P(w, r-1) = \{e(w) - e(m)\}^{r-1} \frac{e(m)\,h(m)}{m}$,

where m is the greatest integer contained in w.

II. INTRODUCTION

In 1951, Mohanty [4; Theorem 3] proved the following:

THEOREM A. *Let* $b > 0$ *and* $c = 1 + \frac{1}{b}$. *Then*

(2.1) $$t^{-b}\varphi(t) \in BV(0, \pi)$$

implies that

$$\Sigma A_n(x) \in |R, e(w), 1|.$$

Generalising the above Theorem Chandra [2] proved the following:

THEOREM B. *Let,*

(2.2) *for* $0 < b < 1$, $c > 0$ *and* $d \geq 0$, $bc = 1 + d$.

Then (2.1) *implies that* $\Sigma A_n(x)\,h(n) \in |R, e(w), 1|$.

The following analogue of Theorem B for conjugate series of the Fourier series is due to Chandra [3]:

THEOREM C. *Let* (2.2) *hold. Then*

(2.3) $$t^{-b}\psi(t) \in BV(0, \pi)$$

implies that

$$\Sigma B_n(x)\,h(n) \in |R, e(w), 1|.$$

The object of this paper is to prove the following theorems which generalise Theorems B and C:

THEOREM 1. *Let* (2.2) *hold. Then* (2.1) *implies that*

$$\Sigma A_n(x)\, h(n) \in |\, R\,, e(w)\,, r\,| \qquad (r > b).$$

THEOREM 2. *Let* (2.2) *hold. Then* (2.3) *implies that*

$$\Sigma B_n(x)\, h(n) \in |\, R\,, e(w)\,, r\,| \qquad (r > b).$$

III. We shall use the following lemmas in the proofs of the theorems:

LEMMA 1. $\Sigma a_n \in |\, R\,, L(w)\,, r\,|$ $(r \geq 0)$ *implies that*

$$\Sigma a_n \in |\, R\,, L(w)\,, r'\,| \qquad (r' > r).$$

This is due to Obrechkoff [5, 6].

LEMMA 2. *Let* (2.2) *hold and let* $0 < r \leq 1$. *Then, uniformly in* $0 < t \leq \pi$ *and* $w \to \infty$,

$$\sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, h(n)\, \frac{\exp(\mathrm{int})}{n} =$$

$$= O\{t^{-r}\, w^{-1}\, e^r(w)\, (\log w)^{d+c(1-r)}\} + P(w\,, r-1)\,.$$

Proof. Let w_1 stand for the integral part of $\left(w - \frac{1}{t}\right)$. Then, we write

$$\sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, h(n)\, \frac{\exp(\mathrm{int})}{n} =$$

$$= \sum_{n=1}^{w_1} + \sum_{1+w_1}^{m} = \Sigma_1 + \Sigma_2\,, \qquad \text{say}.$$

Now for p, the integral part of $e^{2c}\,\Gamma(c+4)$, we write

$$\Sigma_1 = \sum_{n=1}^{p} + \sum_{n=p+1}^{w_1} = O\{e^{r-1}(w)\} + \sum_{n=p+1}^{w_1}\,.$$

Since $(e(w) - e(n))^{r-1} \uparrow$ with n and $\left\{\frac{e(n)\,h(n)}{n}\right\} \uparrow$ with $n > p$, we have

$$\sum_{n=p+1}^{w_1} = \sum_{n=p+1}^{w_1} (e(w) - e(n))^{r-1}\, \frac{e(n)\,h(n)}{n}\, \exp(\mathrm{int}) =$$

$$= O\left\{\left(e(w) - e\left(w - \frac{1}{t}\right)\right)^{r-1} \frac{e\left(w - \frac{1}{t}\right) h(w)}{w - \frac{1}{t}} \max_{1+p \leq p' \leq w_1} \left|\sum_{n=p'}^{w_1} \exp(\mathrm{int})\right|\right\} =$$

$$= O\left\{t^{-1} w^{-1} \left(t^{-1} e^{(1)} \left(w - \frac{1}{t}\right)\right)^{r-1} e^{(1)} \left(w - \frac{1}{t}\right) (\log w)^{c+d}\right\} =$$

$$= O\left\{t^{-r} w^{-1} e^r (w) (\log w)^{d+c(1-r)}\right\},$$

uniformly in $0 < t < \pi$. Therefore, on substituting the order-estimate for $\sum_{n=p+1}^{w_1}$, we get

$$\Sigma_1 = O\left\{t^{-r} w^{-1} e^r (w) (\log w)^{d+c(1-r)}\right\},$$

uniformly in $0 < t < \pi$.

Now

$$\Sigma_2 = O\left\{\int_{w-\frac{1}{t}}^{w} (e(w) - e(y))^{r-1} \frac{e(y)\, h(y)}{y} \, dy\right\} + P(w, r-1) =$$

$$= O\left\{\int_{w-\frac{1}{t}}^{w} (e(w) - e(y))^{r-1} \frac{e^{(1)}(y) (\log y)^{d+c}}{y} \, dy\right\} + P(w, r-1) =$$

$$= O\left\{w^{-1} (\log w)^{d+c} \int_{w-\frac{1}{t}}^{w} (e(w) - e(y))^{r-1} e^{(1)}(y) \, dy\right\} + P(w, r-1) =$$

$$= O\left\{w^{-1} (\log w)^{d+c} \left(e(w) - e\left(w - \frac{1}{t}\right)\right)^r\right\} + P(w, r-1) =$$

$$= O\left\{w^{-1} (\log w)^{d+c} (t^{-1} e^{(1)}(w))^r\right\} + P(w, r-1) =$$

$$= O\left\{t^{-r} w^{-1} e^r (w) (\log w)^{d+c(1-r)}\right\} + P(w, r-1),$$

uniformly in $0 < t \leq \pi$.

On collecting the results for Σ_1 and Σ_2 we follow the proof.

LEMMA 3. *Uniformly in* $0 < t \leq \pi$ *and for* $0 < b \leq 1$

$$\int_0^t u^{b-1} \sin nu \, du = O(n^{-b}) \tag{3.1}$$

and

$$\int_0^t u^b \sin nu \, du = -t^b \frac{\cos nt}{n} + O(n^{-1-b}), \tag{3.2}$$

for large n.

The proof of (3.1) is included in Lemma 2 of Chandra [1] and for the proof of (3.2), see Chandra [3]; (3.2).

IV. Proof of Theorem 1 [(2)]

We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \varphi(t)\cos nt\, dt = \frac{2}{\pi}\pi^{-b}\varphi(\pi)\int_0^{\pi} t^b \cos nt\, dt -$$

$$-\int_0^{\pi} d\{t^{-b}\varphi(t)\}\int_0^{t} u^b \cos nu\, du\,,$$

integrating by parts.

The series $\Sigma A_n(x)\, h(n) \in |R, e(w), r|\ (r > b)$, if

$$I = \int_3^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n) \int_0^t u^b \cos nu\, du \right| dw = O(1)\,,$$

uniformly in $0 < t < \pi$, since by (2.1) $\pi^{-b}\varphi(\pi)$ and $\int_0^{\pi} |d\{t^{-b}\varphi(t)\}|$ are finite.

Integrating the inner integral by parts, we have

$$I \leq t^b \int_3^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n)\, \frac{\sin nt}{n} \right| dw +$$

$$+ b \int_3^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} \frac{e(n)\, h(n)}{n} \int_0^t u^{b-1} \sin nu\, du \right| dw =$$

$$= I_1 + I_2\,, \qquad \text{say.}$$

By (3.1) of Lemma 3, we have

$$P_n = \int_0^t u^{b-1} \sin nu\, du = O(n^{-b})\,,$$

therefore the series $\Sigma \frac{h(n)}{n} P_n \in |R, e(w), 0|$. And, by Lemma 1, the convergence of I_2 follows. Thus, for the proof of Theorem 1, we only require to prove that

$$J = \int_3^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n)\, \frac{\sin nt}{n} \right| dw = O(t^{-b})\,,$$

uniformly in $0 < t < \pi$.

(2) For the proof of Theorems 1 and 2 we take $0 < r \leq 1$ in view of Lemma 1 of the present paper.

For $T = 3 \exp\{t^{-b/(1+d)}\}$, we write

$$J = \int_3^T + \int_T^\infty = J_1 + J_2 , \qquad \text{say.}$$

Since sint $nt = O(1)$, we have

$$J_1 = O\left\{\int_3^T \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} \frac{e(n)\,h(n)}{n} \right| dw\right\} =$$

$$= O\left\{\int_3^T \frac{e^{(1)}(w)}{e^{1+r}(w)}\, dw \int_3^w (e(w) - e(x))^{r-1} \frac{e(z)\,h(z)}{z}\, dz\right\} + O(1) .$$

Now set, for $y > 0$,

$$P_y = \int_3^T \frac{e^{(1)}(w)}{e^{1+r}(w)}\, dw \int_3^w (e(w) + y - e(z))^{r-1} \frac{e(z)\,h(z)}{z}\, dz .$$

Then, by changing the order of integration, we have

$$P_y = \int_3^T \frac{e(z)\,h(z)}{z}\, dz \int_z^T (e(w) + y - e(z))^{r-1} \frac{e^{(1)}(w)}{e^{1+r}(w)}\, dw .$$

And

(4.1) $$\int_3^T \frac{e^{(1)}(w)}{e^{1+r}(w)}\, dw \int_3^w (e(w) - e(z))^{r-1} \frac{e(z)\,h(z)}{z}\, dz = \underset{y\to 0}{Lt}\, P_y =$$

$$= \int_3^T \frac{e(z)\,h(z)}{z}\, dz \int_z^T (e(w) - e(z))^{r-1} \frac{e^{(1)}(w)}{e^{1+r}(w)}\, dw$$

$$= \int_3^T \frac{e(z)\,h(z)}{z}\, Q(z)\, dz . \qquad \text{say.}$$

Integrating by parts, we obtain that

(4.2) $$Q(z) = \left[\frac{(e(w) - e(z))^r}{r e^{1+r}(w)}\right]_z^T + \frac{r+1}{r} \int_z^T \frac{(e(w) - e(z))^r}{e^{2+r}(w)}\, e^{(1)}(w)\, dw =$$

$$= O\{e^{-1}(T)\} + O\left\{\int_z^T \frac{e^{(1)}(w)}{e^{2}(w)}\, dw\right\} = O\{e^{-1}(z)\} .$$

Combining (4.1) and (4.2), we have that

$$J_1 = O\left\{\int_3^T \frac{h(z)}{z}\,dz\right\} + O(1) = O(t^{-b}),$$

uniformly in $0 < t < \pi$.

By Lemma 2, we obtain that

$$J_2 = O\left\{t^{-r}\int_T^\infty \frac{e^{(1)}(w)}{we(w)}(\log w)^{d+c(1-r)}\,dw\right\} + \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)}\,P(w, r-1)\,dw =$$

$$= O\left\{t^{-r}\int_T^\infty \frac{1}{w(\log w)^{cr-d}}\,dw\right\} + \sum_{m=3}^\infty \int_m^{m+1} \frac{e^{(1)}(w)\,P(w, r-1)}{e^{1+r}(w)}\,dw =$$

$$= O\{t^{-r}(\log T)^{d+1-cr}\} + \frac{1}{r}\sum_{m=3}^\infty \frac{e(w)\,h(m)}{me^{1+r}(m)}\left[\{e(w)-e(m)\}^r\right]_m^{m+1} =$$

$$= O\{t^{-r}(\log T)^{c(b-r)}\} + O\left\{\sum_{m=3}^\infty \frac{h(m)}{m}\left(\frac{e^{(1)}(m+1)}{e(m)}\right)^r\right\} \quad \text{(by (2.2))} =$$

$$= O(t^{-b}) + O(1) = O(t^{-b}),$$

uniformly in $0 < t < \pi$.

This completes the proof of Theorem 1.

V. Proof of Theorem 2

We have

$$B_n(x) = \frac{2}{\pi}\int_0^\pi t^{-b}\,\psi(t)\,t^b \sin nt\,dt =$$

$$= \frac{2\,\psi(k)}{\pi^{1+b}}\int_0^\pi u^b \sin nu\,du - \frac{2}{\pi}\int_0^\pi d\{t^{-b}\,\psi(t)\}\int_0^t u^b \sin nu\,du,$$

integrating by parts.

The series $\Sigma B_n(x)\,h(n) \in |R, e(w), r|\ (r > b)$, if

$$J = \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)}\left|\sum_{n<w}(e(w)-e(n))^{r-1}\,e(n)\,B_n(x)\,h(n)\right|\,dw$$

is convergent. Now

$$J \leq \frac{2 \mid \psi(x) \mid}{\pi^{1+b}} \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n) \cdot \right.$$

$$\left. \cdot \int_0^\pi u^b \sin nu \, du \right| dw +$$

$$+ \frac{2}{\pi} \int_0^\pi \mid d\{t^{-b} \psi(t)\} \mid \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} \cdot e(n)\, h(n) \cdot \right.$$

$$\left. \cdot \int_0^t u^b \sin nu \, du \right| dw .$$

Since, by (2.3), $\pi^{-b} \mid \psi(\pi) \mid$ and $\int_0^\pi \mid d(t^{-b} \psi(t)\} \mid$ are finite, therefore, for the proof of the theorem, we only require to prove that

$$I = \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n) \int_0^t u^b \sin nu \, du \right| dw = O(1) ,$$

uniformly in $0 < t \leq \pi$.

Now, by (3.2) of Lemma 3, we have

$$I = O\left\{ t^b \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, h(n) \frac{\cos nt}{n} \right| dw \right\} +$$

$$+ O\left\{ \int_3^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n) \frac{h(n)}{n^{1+b}} \right| dw \right\} =$$

$$= O(I_1) + O(I_2) , \qquad \text{say.}$$

The convergence of I_2 follows from Lemma 1 since $\Sigma \frac{h(n)}{n^{1+b}} \in \mid R, e(w), 0 \mid$. The uniform boundedness of I_1, in $0 < t \leq \pi$, runs parallel to that of I_1 of Theorem 1 of this paper.

This terminates the proof of Theorem 2.

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