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Tallini sets in projective spaces

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Geometria. — *Tallini sets in projective spaces.* Nota (*) di CHRISTIANE LEFÈVRE (**), presentata dal Socio B. SEGRE.

RIASSUNTO. — Vengono studiati i sottoinsiemi di uno spazio proiettivo che da ogni retta dello spazio, che ad essi non appartenga per intero, siano incontrati in non più di due punti.

1. INTRODUCTION

One of the most beautiful results in finite geometry is Segre's characterization of conics in projective planes $P_2(q)$, with q odd, as maximal sets of points meeting each line in at most two points [6]. In view of this theorem, it was natural to attempt the same approach for higher dimensional quadrics (see for instance [1], [5]). A very important work in this direction is due to Tallini [9]: he obtained a characterization of the hyperbolic quadrics in the n-dimensional projective spaces $P_n(q)$, with n odd, $q > 2$, and of the quadrics in $P_n(q)$, with n even, $q > 2$. To this end, he classified all sets of points of cardinality at least $q^{n-1} + q^{n-2} + \cdots + 1$ in $P_n(q)$, such that each line intersects them in $0, 1, 2$ or $q + 1$ points. The bound on the number of points is fundamental in the proof of this classification, and so Tallini had to give [10] a particular characterization of the elliptic quadrics, using especially the exact number of their points (which is inferior to the bound taken above).

In recognition of this work, we define a *Tallini set* as a set Q of points of a projective space such that each line, not contained in Q, intersects it in at most two points.

The purpose of this paper is to investigate Tallini sets with no finiteness assumption on the projective space. A complete classification of Tallini sets is given in projective spaces of dimension $n \leq 4$. Furthermore, constructions of infinite families of Tallini sets are obtained in any dimension. These constructions show that the class of Tallini sets in any projective space is very large and so we should expect that their complete classification is hopeless.

In two other papers [3], [4], we treat of another fundamental question, namely the characterization of quadrics as Tallini sets in finite and infinite projective spaces. However we shall prove here a negative result in this direction: complete (i.e. maximal in the set-theoretical sense) Tallini sets in a projective space P are not necessarily quadrics in P.

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2. DEFINITIONS AND EXAMPLES

First of all, we introduce some definitions and notations.

Let P be a projective space. If p, q are points of P, then the line through p and q is denoted by pq and if X is any subset of P, the subspace of P genera-

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ted by X is denoted by $\langle X \rangle$. A *Tallini set in* P is a set Q of points of P such that:

(1) For every line L of P, not contained in Q, we have $|L \cap Q| \leq 2$.

(2) $\langle Q \rangle = P$.

A line L which in contained in Q is called a *line of* Q. Two points $p, q \in Q$ are said to be *adjacent*, and we write $p \sim q$, if the line pq is a line of Q. For convenience we say that p is adjacent to itself. A point $p \in Q$ is a *double point of* Q if p is adjacent to all points of Q and Q is called *degenerate* if Q has some double point. Notice that if S is a subspace of P, then the set $S \cap Q$ is a Tallini set in a (possibly proper) subspace of S.

If Q does not contain any line, we say, according to a standard terminology [8], that Q is a *cap* in P or an *arc* in P, if P is a projective plane. Well known results show that a classification of caps in finite projective spaces is hopeless. Then, a classification of Tallini sets containing points through which there is no line of Q is also hopeless. Therefore we shall often (Sections 3.4, 4 and 6) restrict ourself to the study of Tallini sets which are union of lines and we shall call them *ruled Tallini sets*.

Quadrics are the main Tallini sets. Other examples are ovoids and all subsets of them. If P has order 2, the situation is trivial: all subsets of P (generating P) are Tallini sets in P. Consequently, we shall always assume that *P is a projective space whose lines have at least 4 points*. Finally, we observe that Tallini sets are much more general than quadratic sets [2], the difference being that we drop the tangent hyperplanes in the definition of Tallini sets.

3. CLASSIFICATION OF TALLINI SETS IN PROJECTIVE SPACES OF DIMENSION $n \leq 4$

In this section, we give a list of all Tallini sets in projective spaces of dimension $n \leq 3$ and of non degenerate ruled Tallini sets in dimension $n = 4$. This classification is obtained through elementary geometric arguments. In dimension $n = 2$, the result is immediate; we shall give a complete proof for $n = 3$; the case $n = 4$ leads to an analysis of many cases and so we only indicate the main steps of our proof.

3.1. *The 1-dimensional case.*

Every Tallini set in a projective line P_1 is either two points of P_1, or the line P_1 itself.

3.2. *The 2-dimensional case.*

Every Tallini set in a projective plane P_2 is one of the following:

(i) an arc in P_2 (in particular a non degenerate conic);

(*ii*) the union of a line L of P_2 and a subspace of dimension 0 or 1 in P_2, not contained in L;

(*iii*) the plane P_2 itself.

3.3. *The 3-dimensional case.*

PROPOSITION 1. *Every Tallini set* Q *in a 3-dimensional projective space is one of the following*:

(*i*) *a cap of* P_3 (*in particular an elliptic quadric*);

(*ii*) *the union of a subset* (*generating* P_3) *of the following set of lines of* P_3: *two skew lines* L_1 *and* L_2 *and two other skew lines* M_1 *and* M_2 *intersecting* L_1 *and* L_2;

(*iii*) *a ruled quadric in* P_3;

(*iv*) *the union of lines* L_i *joining a point* p *of* P_3 *to an arc in some plane exterior to* p, *together with a* (*possibly empty*) *set* K, *which is a cap in* $\langle K \rangle$, *no point of* K *being contained in a plane* $\langle L_i, L_j \rangle$ *and no pair of points of* K *being coplanar with a line* L_i;

(*v*) *the union of a plane* π *and a subspace of dimension* 0, 1 *or* 2 *in* P_3, *not contained in* π;

(*vi*) *the space* P_3 *itself.*

Proof. Case (*i*) occurs whenever the set of lines of Q is empty. Otherwise, we distinguish the three following cases.

1) *Q contains two skew lines* L_1 *and* L_2 *but no plane of* P_3.

As the set $L_1 \cup L_2$ is a Tallini set in P_3, Q may of course consist of $L_1 \cup L_2$, a Tallini set of type (*ii*). Suppose now there exists a point p_1 of Q not on L_1 nor on L_2. Then the unique line M_1 through p_1 intersecting L_1 and L_2 must be a line of Q. The set $L_1 \cup L_2 \cup M_1$ is another Tallini set in P_3 of type (*ii*). Now if there exist in Q three lines L_1, L_2, M_1 as above and a point $p_2 \notin L_1, L_2, M_1$, then the line M_2 through p_2 intersecting L_1 and L_2 must also be a line of Q. This line cannot intersect M_1, otherwise there would be three lines in a plane and, by the 2-dimensional classification, this plane would be in Q, a contradiction. The union $L_1 \cup L_2 \cup M_1 \cup M_2$ is a Tallini set in P_3 and we get so the last Tallini set of type (*ii*).

Finally if there exists in Q a configuration of four lines L_1, L_2, M_1, M_2 as above, together with a point $p_2 \notin L_1, L_2, M_1, M_2$, then the lines L_3 and M_3 through p_3 intersecting respectively M_1, M_2 and L_1, L_2 are in Q. These lines L_3 and M_3 must be respectively skew to L_1, L_2 and M_2, M_3. Consequently, all lines intersecting L_1, L_2, L_3 (a regulus $\mathscr{M}$) as well as all lines intersecting M_1, M_2, M_3 (a secnd regulus $\mathscr{L}$) have to be in Q. By [7]. the sets of points of these two reguli coincide if and only if the ground field of P_3 is commutative and then this set of points is a ruled quadric in P_3, which is of course a Tallini set in P_3 (type (*iii*)). In the non commutative case, consider the regulus $\mathscr{L}$ and a point p on a line of $\mathscr{M}$ but on no line of $\mathscr{L}$. Then $p \notin M_1, M_2$ nor M_3 by the definition of $\mathscr{L}$. The line N through p intersecting M_1 and M_2 is in Q. But through each point of M_1, there exists a line $L_i \in \mathscr{L}$ intersecting M_2. Hence, as Q does not contain any plane, N must be a line L_i, a contradiction to the choice of p. Hence there is no Tallini set, satisfying to 1), containing $\mathscr{L}$ and $\mathscr{M}$, in the non commutative case.

As it is clear that there is no Tallini set in P_3 containing properly a ruled quadric, except P_3 itself, we can conclude that 1) leads exactly to cases (*ii*) and (*iii*).

2) *All lines of* Q *intersect in one point* p.

It is clear that the set of lines of Q is the set of lines joining p to the points of an arc in a plane exterior to p. As Q may contain points on no line, this leads to case (*iv*).

3) *Q contains a plane* π *of* P_3.

As $(Q) = P_3$, there exists a point p of Q not in π and Q may of course be $\pi \cup \{p\}$, which is a Tallini set in P_3. Now if Q contains π and two distinct points p_1 and p_2 not in π, then Q contains the line $p_1 p_2$. As the union $\pi \cup p_1 p_2$ is a Tallini set in P_3, Q may consist of $\pi \cup p_1 p_2$. But if Q contains three non collinear points p_1, p_2, p_3 not in π, consider the plane $\pi' = \langle p_1, p_2, p_3 \rangle$. The Tallini set $\pi' \cap Q$ contains the line $\pi' \cap \pi$ and the non collinear points p_1, p_2, p_3 which are not in π. Hence, by the classification of Tallini sets in a plane, π' is in Q and Q contains the union $\pi \cup \pi'$, which is a Tallini set in P_3. It is trivial that, if Q contains a point exterior to two planes of P_3, then Q is P_3 itself. Consequently, we have proved that 3) leads to cases (*v*) and (*vi*) and so the proof of Proposition 1 is complete.

3.4. *The 4-dimensional case.*

A complete list of all 4-dimensional Tallini sets would be too long to be stated here. As we noticed in Section 2, it is reasonable to restrict ourself to the study of ruled Tallini sets of P_4. Furthermore, by the result which we shall establish in Section 4, the classification of degenerate Tallini sets of P_4 is reduced to the classification of Tallini sets in lower dimensions. Consequently, we give here the list of all 4-dimensional non degenerate ruled Tallini sets.

PROPOSITION 2. *Every non degenerate ruled Tallini set* Q *in a 4-dimensional projective space* P *is one of the following*:

(*i*) *a quadric in* P_4;

(*ii*) *the union of two Tallini sets* Q' *and* Q'', *respectively in proper subspaces* P' *and* P'' *of* P_4 *such that* $\langle P', P'' \rangle = P_4$ and $Q' \cap P'' = Q'' \cap P'$;

(*iii*) *the union of lines* L_i *through a point* p *and lines* M_i *through a point* q, *such that no three of them are coplanar and, for every two skew lines* L_i *and* M_j, *the subspace* $\langle L_i, M_j \rangle$ *contains at most two other lines* L_k *and* M_l, *where* L_k *intersects* M_j *and* M_l *intersects* L_i;

(*iv*) *the union of a line* L *and skew lines* L_i *intersecting* L, *such that the planes* $\langle L, L_i \rangle$ *meet a plane* π *disjoint from* L *in points of an arc* K *of* π;

(*v*) *the union of a Tallini set of the preceeding type, together with lines* M_i, *such that the planes* $\langle L, M_i \rangle$ *intersect* π *in points which are on no secant nor tangent to* K, *every three of these points being collinear if and only if they correspond to lines* L_i *intersecting* L *at the same point*;

(*vi*) *the union of a 3-dimensional ruled quadric* Q' *together with a set of lines joining a point* $p \notin \langle Q' \rangle$ *to points of* Q' *which are pariwise non adjacent in* Q';

(*vii*) *the union of a plane* π *of* P_4, *a line* L *disjoint from* π *and skew lines* L_i *intersecting* L *and meeting* π *in points of an arc* K *of* π;

(*viii*) *the union of a plane* π *of* P_4 *and lines* L_i *intersecting* π, *no three of these lines being coplanar, no two of them intersecting outside* π, *no subspace* $\{L_i, L_j\}$ *containing* π *and such that three of the lines* L_i *are contained in the same hyperplane if and only if they intersect* π *at the same point*;

(*ix*) *the union of planes* π_i *through a line* L *and lines* L_i *intersecting* L, *such that a hyperplane not containing* L *intersects* π_i *and* L_i in *a* 3-*dimensional Tallini set of type* (iv), *where every three points of* K *are coplanar with* p *if and only if they correspond to lines* L_i *which intersect* L *at the same point*;

(*x*) *the space* P_4 *itself*.

MAIN STEPS OF A PROOF. – We suppose case (x) does not occur. It is easy to see that there does not exist any degenerate ruled Tallini set containing a hyperplane. Hence, suppose Q contains a plane but no hyperplane of P_4. Then we prove that, if there exists a line L skew to π, Q is of type (*vii*) or contains a 3-dimensional ruled quadric; in the latter case, Q must be of type (*ii*). If there is no line skew to π, then (*viii*) and (*ix*) are realized. Now, if Q does not contain any plane we distinguish two cases:

1) Q contains a 3-dimensional ruled quadric. Then, we prove that Q must be of type (*ii*), unless (*vi*) is realized.

2) Q contains no 3-dimensional ruled quadric. Then, it is possible to show that we may suppose the existence of two hyperplanes H_1 and H_2 intersecting Q in the configuration of four lines described in (*iv*) of 3.3. Following the "nature" of the Tallini set $H_1 \cap H_2 \cap Q$, we obtain various possibilities leading to (*ii*), (*iii*), (*iv*) or (*v*).

4. DEGENERATE TALLINI SETS

In this section, we show that every degenerate Tallini set can be described in terms of a non degenerate one. Consequently the problem of classifying all Tallini sets is reduced to the classification of all non degenerate ones.

LEMMA. *Let* Q *be a Tallini set in a projective space* P. *Then the set* A *of double points of* Q *is a subspace of* P.

Proof. We have to prove that, if a and b are two double points of Q, then all points c of the line ab are double points of Q. As a (or b) is a double point, a is adjacent to b and so the line ab is in Q. Hence $c \in Q$ and c is adjacent to all points of ab. Now let p be any point of $Q - ab$ and consider the plane $\langle p, a, b\rangle$. As a and b are double points, the intersection $\langle p, a, b\rangle \cap Q$ contains the three lines pa, pb and ab. Hence the plane $\langle p, a, b\rangle$ must be in Q and so all points c of ab are adjacent to p and the lemma is proved.

The following proposition gives a description of degenerate Tallini sets.

PROPOSITION 3. *Let* Q *be a Tallini set in a projective space* P *and let* A *be the set of its double points. If* B *is a complementary subspace of* A *in* P [1], *then* $B \cap Q$ *is a non degenerate Tallini set in* B *and* Q *is the union of all lines joining a point of* A *to a point of* $B \cap Q$.

(1) B is a complementary subspace of A in P if $A \cap B = \varnothing$ and $\langle A, B\rangle = P$.

Proof. First of all, we show that Q is the union of all lines ab, for $a \in A$ and $b \in B \cap Q$. As a is a double point of Q, the lines ab, where $a \in A$ and $b \in B \cap Q$, are lines of Q. Now, if p is a point of Q — (A ∪ B), the lines ap, for $a \in A$, are in Q. But the union of these lines is the subspace $\langle A, p \rangle$ which intersects B at point $\bar{b}$. Hence the point $\bar{b}$ is in B ∩ Q and so p is the union of all lines ab, where $a \in A$ and $b \in B \cap Q$. We prove now that B ∩ Q is a non degenerate Tallini set in *B*. The fact that B ∩ Q is a Tallini set (perhaps in a proper subspace of B) is trivial (see Section 2). As $\langle Q \rangle = P$, the result proved here above shows that B ∩ Q is a Tallini set in B. We have to show that B ∩ Q is non degenerate. Suppose, by way of contradiction, that x is a double point of B ∩ Q. We shall prove that x is a double point of Q, which contradicts the hypothesis A ∩ B empty. We have to prove that each point p of Q is adjacent to x. Consider the point $b = B \cap \langle A, p \rangle$. If $b = x$, then $p \sim x$ because $\langle A, p \rangle \subset Q$ (see the first part of this proof). If $b \neq x$, then the plane $\langle a, p, x \rangle$ coincides with the plane $\langle a, b, x \rangle$. The latter is contained in Q, as a is a double point of Q and $b \sim x$. Hence $p \sim x$ and the proof is complete.

Now we show that every set described in Proposition 3 is a Tallini set in P and so we prove that the class of all degenerate Tallini sets in a projective space P is determined by the family of non degenerate Tallini sets in proper subspaces of P.

PROPOSITION 4. *Let* A *and* B *be two complementary subspaces of a projective space* P *and let* $\bar{Q}$ *be a non degenerate Tallini set in* B. *Then the union* Q *of all lines joining every point of* A *to all points of* $\bar{Q}$ *is a Tallini set in* P, *with* A *as set of double points.*

Proof. Condition (2) of Section 2 is clearly satisfied because $\langle A, \bar{Q} \rangle = P$. We shall show that each line L of P, not contained in Q, intersects Q in at most two points. If L intersects A at a point a, then L cannot intersect Q in a point distinct from a, otherwise by the definition of Q, L would be in Q. If L does not intersect A, then the space $\langle A, L \rangle$ intersects B in a line. This one cannot belong to $\bar{Q}$, because L is not in Q. Hence suppose, by way of contradiction, that L ∩ Q contains three distinct points p_1, p_2, P_3. Let b_1, b_2, b_3 be the points of $\bar{Q}$ such that $p_i \in ab_i$, for all $a \in A$; the points b_i are distinct because L and A are disjoint. Then it is clear that $b_1, b_2, b_3 \in \langle A, L \rangle \cap B$ and so there exists a line, not belonging to $\bar{Q}$, intersecting it in more than two points, which is a contradiction. Consequently, we have proved that Q is a Tallini set. Furthermore it is clear that A is the subspace of double points of Q.

5. CONSTRUCTION OF TALLINI SETS

As we said in the Introduction, examples of Tallini sets in a projective space P are given by the quadrics in P. But some Tallini sets described in Section 3 show that, more generally, suitable unions of quadrics are also Tallini sets in P. This remark gives rise to the following general construction.

CONSTRUCTION 1. *Consider a family (finite or not) of pairs* $(P_i, Q_i)_{i \in I}$, *where* I *is a well ordered set,* P_i *a proper subspace of* P *and* Q_i *a quadric in* P_i (i.e. *generating* P_i) *such that*:

(*a*) $\langle \bigcup_{i \in I} P_i \rangle = P$;

(*b*) *For all k, the subspace* P_k *is not contained in the subspace* $\langle \bigcup_{i<k} P_i \rangle$;

(*c*) *For all, k the intersection* $Q_k \cap \langle \bigcup_{i<k} P_i \rangle$ *coincides with the intersection* $P_k \cap (\bigcup_{i<k} Q_i)$.

Then the union $Q = \bigcup_{i \in I} Q_i$ *is a Tallini set in* P.

Proof. By (*a*), condition (2) of Section 2 is satisfied. As for (1), consider a line L of P, not contained in Q. If L is contained in one of the P_i's, this line L intersects of course Q in at most two points. Suppose that L is not contained in a subspace P_i; we have to prove that $|L \cap Q| = 2$, whenever L is joining two points of Q. Denote these two points by p_j and p_k, where j and k are the minimum indices in I such that the quadrics Q_j and Q_k contain respectively p_j and p_k. Furthermore suppose $j < k$. Then, as $j < k$, the line L is contained in the subspace $\langle \bigcup_{i \leq k} P_i \rangle$ and, as j and k are minimal, L intersects $\langle \bigcup_{i<k} P_i \rangle$ exactly in p_j (see condition (*b*)). But $Q \cap \langle \bigcup_{i \leq k} Q_i \rangle$ is $\bigcup_{i \leq k} Q_i$. Hence $L \cap Q = L \cap (\bigcup_{i \leq k} Q_i) = \{p_j, p_k\}$ and the proof is finished.

Remarks. 1. The Tallini set Q described above may be the union of two Tallini sets in disjoint subspaces. To avoid these "trivial" Tallini sets, we shall essentially consider here Tallini sets which are *connected by lines*, i.e. satisfying the following condition: for every two points p, q of Q, there exists a sequence of lines $L_1, \cdots, L_k$ such that (*i*) $p \in L_1, q \in L_k$ and (*ii*) $|L_i \cap L_{i-1}| = 1$ for all i.

2. Construction 1 allows to prove the following property. *If* P_0 *is a proper subspace of* P *and* Q_0 *a Tallini set in* P_0, *then there exists a Tallini set* Q *in* P *containing* Q_0 *and being connected by lines*. Indeed, let $\overline{Q}$ be a quadric (possibly degenerate or reduced to a point) contained in Q_0 and let P_1 be a subspace of P intersecting P_0 in $\langle \overline{Q} \rangle$. Then, following Construction 1, we can build families of quadrics $\{Q_i\}_{i \in I}$ in subspaces $\{P_i\}_{i \in I}$ such that $Q_0 \cup (\bigcup_{i \in I} Q_i)$ are Tallini sets Q in P, containing Q_0. These Tallini sets are connected if we choose the subspace P_i such that, for all k, $Q_k \cup (\bigcup_{i \in I} P_i)$ is non empty.

3. From this second remark, we get an immediate generalization of Construction 1 by considering *pairs* $(P_i, Q_i)_{i \in I}$ *with* Q_i *a Tallini set in* P_i, *instead of a quadric*.

4. Finally, if I is finite, a Tallini set obtained by the latter construction is the union of two Tallini sets Q_1^* and Q_2^* in proper subspaces P_1^* and P_2^*, such that $\langle P_1^*, P_2^* \rangle = P$ and $Q_1^* \cap P_2^* = Q_2^* \cap P_1^*$. Indeed, we can take $Q_1^* = \bigcup_{i \leq l} Q_i$ and $Q_2^* = \bigcup_{i > l} Q_i$ for some $l \in I$; then $P_1^* = \langle \bigcup_{i \leq l} P_i \rangle$ and $P_2^* = \langle \bigcup_{i > l} P_i^* \rangle$.

We give now another construction, showing that Tallini sets are not necessarily of the preceeding kind.

CONSTRUCTION 2. *Let* A *and* B *be two complementary subspaces of* P. *Let* Q_0 *be a Tallini set in* A *and* $K = \{p_i\}_{i \in I}$ *be a cap in* B. *Consider the subspaces* $P_i = \langle A, p_i \rangle$, *for* $i \in I$, *and let* Q_i *be a Tallini set in* P_i *such that* $Q_i \cap A = Q_0$ *Then* $Q = \bigcup_{i \in I} Q_i$ *is a Tallini set in* P *and* Q *is connected by lines if and only if the* Q_i's *are connected by lines.*

Proof. First of all, let us note that the existence of the Tallini sets Q_i is a consequence of Remark 2. We shall prove that Q is a Tallini set. Condition (2) is satisfied. Now, consider a line L not in Q. If L is in P_i, then $|L \cap Q| = |L \cap Q_i| \leq 2$. If L is not contained in a subspace P_i, then L meets exactly two subspaces P_i, otherwise K would not be a cap. Hence L intersects $Q = \bigcup_{i \in I} Q_i$ in at most two points and conditions (1) is proved.

Note that if K consists of two points, Construction 2 is the same as the construction given in Remark 3. Finally, remark that cases (*iv*) of the 3 and 4-dimensional classifications of Tallini sets are obtained by this construction.

6. COMPLETE TALLINI SETS

If we want to characterize quadrics among Tallini sets, the question arises whether complete Tallini sets in P, i.e. Tallini sets which are not contained properly in bigger Tallini sets different from P, are quadrics. The answer is negative as it si shown by known caps, but we may state the same question in the class of ruled Tallini sets or even of Tallini sets which are connected by lines. We shall show that the answer is still negative. A counter example arises from the classification of the 4-dimensional Tallini sets. This counter example can be generalized in projective spaces of every finite dimension, by Construction 2 of Section 5.

PROPOSITION 5. *Let* P *be a projective space of finite dimension* $n \geq 4$. *Then there exists in* P *a Tallini set* Q *connected by lines which is not contained in any quadric* $\bar{Q}$ *of* P ($\bar{Q} \neq P$).

Proof. This result is obtained by an induction on the dimension of the projective space.

1) First of all, we prove the proposition for $n = 4$. Consider a Tallini set Q of type (*iv*) mentioned in Section 3.4, i.e. a Tallini set obtained by Construction 2 with $A = Q_0 = L$ and K an arc in a plane π disjoint from L. This Tallini set Q is connected by lines. We suppose that K is not contained in a conic and has cardinality greater than 3. Furthermore, assume that two of the lines L_i, say L_k and L_l, do not intersect π. Then we shall show that Q has the required property. By way of contradiction, let $\bar{Q}$ be a quadric containing Q.

As K is not included in any conic, the plane π must be contained entirely in $\bar{Q}$. But a non degenerate 4-dimensional quadric has index 2[(2)]. Hence $\bar{Q}$ must be degenerate and let D be its space of double points. We show that

(2) Index i means that all maximal subspaces of P contained in Q have dimension $i - 1$.

K is in π. If this were not the case, $\bar{Q}$ would contain a hyperplane through π and so $\bar{Q}$ would be the union of two hyperplanes. But it is easy to see that the union of such two hyperplanes can at most contain three lines L_i of the Tallini set Q. This gives rise to a contradiction because, as $|K| > 3$, Q contains more than three lines L_i and so Q cannot be contained in $\bar{Q}$. Now, the subspace D cannot be the plane π itself (otherwise, as $\bar{Q} \supset Q \supset L$, $\bar{Q}$ would be P itself) nor a line M of π (otherwise $\bar{Q}$ would contain the hyperplane $\langle L, M \rangle$ together with the two other hyperplanes $\langle L_k, M \rangle$ and $\langle L_l, M \rangle$, a contradiction). Hence $\bar{Q}$ has one double point p in π.

Consider now the plane $\langle L, L_k \rangle$. This plane contains the two distinct lines L and L_k, which must be lines of $\bar{Q}$, together with the point $\langle L, L_k \rangle \cap \pi$, which is in $\bar{Q}$ too and is not on L nor L_k. Hence $\langle L, L_k \rangle$ must be in $\bar{Q}$ and so the point $\langle L, L_k \rangle \cap \pi$ must be the double point p of $\bar{Q}$. But the same arguments are valid for the plane $\langle L, L_l \rangle$ and so $\langle L, L_k \rangle \cap \pi = \langle L, L_l \rangle \cap \pi$, i.e. $\langle L, L_k \rangle = \langle L, L_l \rangle$, a contradiction to the definition of Q. This ends the proof for $n = 4$.

2) Suppose now Proposition 5 is valid in dimension $n - i$ $(i > 0)$; we shall prove that it is also valid in dimension n. Let A and B be two complementary subspaces of P having distinct dimensions $n - i$ and $i - 1$ (3). Let Q be a Tallini set obtained by Construction 2, with K a cap in B not contained in a quadric of B (except if B has dimension 0) and with Q_0 a Tallini set in A which is not contained in a quadric of A. The existence of Q_0 in A is a consequence of the induction assumption and of the possible choice $i = 1$: A has dimension $n - i$, with $4 \leq n - i < n$. Then clearly Q is not contained in any quadric $\bar{Q}$, otherwise $\bar{Q}$ would contain A and B, which is impossibile because A and B are complementary subspaces of distinct dimensions.

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(3) B may have dimension 0. This is useful to obtain the result in dimension $n = 5, 6$.