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Compact operators on spaces of integrable functions on homogeneous spaces

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Matematica. — *Compact operators on spaces of integrable functions on homogeneous spaces.* Nota di OLUSOLA AKINYELE, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Vengono caratterizzate alcune classi di operatori compatti su certi spazi di funzioni integrabili su spazi omogenei.

INTRODUCTION

Let S be a k-dimensional sphere ($k \geq 2$) and SO ($k+1$) be the special orthogonal group of rotations on S. If SO (k) is the closed subgroup of SO ($k+1$) leaving a point of S fixed then it is well-known that $S \simeq SO(k+1)/SO(k)$ and S is a special compact homogeneous space. Let L′ (S) be the Banach space of integrable functions on S and $L'(S\,;p) = \{f \in L'(S) : R_\alpha f = f, \alpha \in SO(k+1)\}$ where p is the north pole of S and R_α is the rotation operator defined by $R_\alpha f(x) = f(x\alpha)\ x \in S$; then L′ (S ; p) is a closed Banach subalgebra of L′ (S). In section 2, we characterize the compact operators on L′ (S) which commute with the rotation operators $R_\alpha\ \alpha \in SO(k+1)$ as convolution with elements of L′ (S , p). A similar characterization is also obtained for compact operators on C (S), the space of continuous functions on S. In section 3, we consider a general type of homogeneous space namely; $G|_H$ where G is a compact group and H is closed subgroup of G. We define

$$L'_H(G) = \{f \in L'(G) : f(hx) = h(x)\,,\, x \in G\,,\, h \in H\}$$

and $L'_{HH}(G) = \{f \in L'(G) : f(h \times h') = f(x)\,,\, h \in H\,,\, x \in G\}$. It is known that $L'(G|_H) \simeq L'_H(G)$ [cfr., 3]. We characterize in section 3, the compact operators which are G-operators on $L'_H(G)$ as convolution with elements of $L'_{HH}(G)$. A similar characterization is obtained for $C_H(G) \simeq C(G|_H)$. The results of this paper thus generalize the result of [1] for compact groups to compact homogeneous spaces. For harmonic analysis on homogeneous spaces we use [3] as our main reference.

§ 2. COMPACT OPERATORS ON L′ (S)

We state and prove the main result of this section:

THEOREM 2.1. *Let* $T : L'(S) \to L'(S)$ *be a bounded linear operator which commutes with the rotation operators on* S. *Then* T *is a compact operator if and only if there exists* $g \in L'(S\,;p)$ *such that for each* $f \in L'(S)$, $Tf = g * f$.

(*) Nella seduta del 13 dicembre 1975.

Proof. Assume that $g \in \mathrm{L}'(\mathrm{S}\,;p)$, then given $\varepsilon > 0\ \exists\, \psi_p^{-1}\,\mathrm{P}_m^\lambda, \cdots, m = 0, 1, 2, \cdots, \mathrm{N} \in \mathrm{C}(\mathrm{S}\,;p) \ni \left\| g - \sum_{m=0}^{\mathrm{N}} \alpha_m\, \psi_p^{-1}\, \mathrm{P}_m^\lambda \right\| < \varepsilon$ [cfr., 2].

Hence for $f \in \mathrm{L}'(\mathrm{S})$, assume that $\mathrm{T}f = g * f$, and choose $\varepsilon < \frac{\eta}{\|f\|_1}$ then (cfr. 2],

$$\left\| \mathrm{T}f - \sum_{m=0}^{\mathrm{N}} \alpha_m \tilde{f}_m \right\|_1 = \int_{\mathrm{S}} \left| g * f(x) - \sum_{m=0}^{\mathrm{N}} \alpha_m \tilde{f}_m(x) \right| \mathrm{d}x =$$

$$= \int_{\mathrm{S}} \left| \int_{\mathrm{S}} f(y)\, \varphi_x g(y)\, \mathrm{d}y - \sum_{m=0}^{\mathrm{N}} \alpha_m \int_{\mathrm{S}} f(y)\, \mathrm{P}_m^\lambda(x \cdot y)\, \mathrm{d}y \right| \mathrm{d}x =$$

$$= \int_{\mathrm{S}} \left| \int_{\mathrm{S}} f(y)\, \varphi_x g(y)\, \mathrm{d}y - \int_{\mathrm{S}} f(y) \sum_{m=0}^{\mathrm{N}} \alpha_m\, \mathrm{P}_m^\lambda(x \cdot y)\, \mathrm{d}y \right| \mathrm{d}x =$$

$$= \int_{\mathrm{S}} |f(y)| \int_{\mathrm{S}} \left| \varphi_x g(y) - \sum_{m=0}^{\mathrm{N}} \alpha_m\, \mathrm{P}_m^\lambda(x \cdot y) \right| \mathrm{d}y\, \mathrm{d}x =$$

$$= \int_{\mathrm{S}} |f(y)| \int_{\mathrm{S}} \left| \varphi y g(x) - \sum_{m=0}^{\mathrm{N}} \alpha_m\, \varphi_y\, \mathrm{P}_m^\lambda(x \cdot p) \right| \mathrm{d}y\, \mathrm{d}x =$$

$$= \int_{\mathrm{S}} |f(y)|\, \mathrm{d}y \left\| g - \sum_{m=0}^{\mathrm{N}} \alpha_m\, \psi_p^{-1}\, \mathrm{P}_m^\lambda \right\|_1 \leq \varepsilon \|f\|_1 < \eta\,.$$

Thus T is an operator of finite rank and hence compact.

Assume now that T is compact, then by [2] there exists $\mu \in \mathrm{M}(\mathrm{S}\,;p)$ such that $\mathrm{T}f = f * \mu$ for each $f \in \mathrm{L}'(\mathrm{S})$. Let U_v be an approximate identity in $\mathrm{L}'(\mathrm{S}\,;p)$, then for $g \in \mathrm{C}(\mathrm{S})$, $\mathrm{U}_v * g \to g$ in the norm topology of $\mathrm{C}(\mathrm{S})$ and in particular for $g \in \mathrm{C}(\mathrm{S}\,;p)$, $\mathrm{U}_v * g \to g$. For $\mu \in \mathrm{M}(\mathrm{S}\,;p)$, $\mu * \mathrm{U}_v \in \mathrm{L}'(\mathrm{S}\,;p)$ [2] and $\mathrm{TU}_v = \mu * \mathrm{U}_v$ so that $\|\mathrm{TU}_v\| < \infty$. Hence $\{\mathrm{TU}_v\}$ is a bounded net in $\mathrm{M}(\mathrm{S}\,;p)$ and there exists $\nu \in \mathrm{M}(\mathrm{S}\,;p)$ such that $\mu * \mathrm{U}_v \to \nu$ in the weak* topology, that is if, $f \in \mathrm{C}(\mathrm{S}\,;p)$

$$\int_{\mathrm{S}} f(x)\, \mathrm{d}\nu(x) = \int_{\mathrm{S}} f(x)\, \mathrm{d}(\mu * \mathrm{U}_v)(x) = \int_{\mathrm{S}} f(x) \left\{ \int_{\mathrm{S}} \mathrm{U}_v(y)\, \mathrm{d}\varphi_x\, \mu(x) \right\} \mathrm{d}x.$$

Since $\mu \in \mathrm{M}(\mathrm{S}\,;p)$, let $\alpha \in \mathrm{SO}(k+1)$ such that $\alpha p = p$, then

$$\int_{\mathrm{S}} f(x)\, \mathrm{d}\nu(x) = \int_{\mathrm{S}} f(x) \left\{ \int_{\mathrm{S}} \mathrm{U}_v(y)\, \mathrm{dR}_\alpha\, \mu(y) \right\} \mathrm{d}x =$$

$$= \int_{\mathrm{S}} f(x) \left\{ \int_{\mathrm{S}} \mathrm{U}_v(x)\, \mathrm{d}\mu(x) \right\} \mathrm{d}(x) = \int_{\mathrm{S}} f(x)\, \mathrm{d}\mu(x).$$

So $\nu = \mu$ a.e. Since $\{\mu * U_v\}$ is a bounded net in $L'(S\,;p)$ then there exists a subnet which converges to some $g \in L'(S\,;p)$ in the L'-norm. The convergence of $\mu * U_v$ to μ in the weak*-topology implies that $g = \mu$ and $Tf = g * f$ for all $f \in L'(S)$.

COROLLARY 2.2. *Let* $T : C(S) \to C(S)$ *be a bounded linear operator which commutes with the rotation operator on* S. T *is a compact operator if and only if there exists a* $g \in L'(S\,;p)$ *such that* $Tf = g * f$ *for all* $f \in C(S)$.

Proof. If $\exists g \in L'(S\,;p)$ such that $Tf = g * f$ for all $f \in C(S)$, a similar argument to the first part of Theorem 2.1 shows that T is compact. Assume that T is compact, then by [2], $\exists \mu \in M(S\,;p)$ such that $Tf = \mu * f$ for all $f \in C(S)$. Proceeding in a similar way as in the theorem, $\mu \in L'(S\,;p)$.

§ 3. COMPACT OPERATORS ON $L'(G|_H)$.

In this section G is a compact group and H a closed subgroup of G. Define $C_{HH}(G) = \{f \in C(G) : f(h \times h') = f(x)\,,\, x \in G\,,\, h\ h' \in H\}$ and $C_H(G) = \{f \in C(G) : f(hx) = f(x)\,,\, x \in C\,,\, h \in H\}$ where $C(G)$ is the Banach algebra of continuous functions on G. Let the Haar measure of H be m_H and denote by $\hat{G}$ the dual of G. For $\alpha \in \hat{G}$, let T_α be some element of the equivalent class of continuous unitary irreducible representation of G denoted by α, and define $\chi_\alpha(x) = T_r(T_\alpha(x)) = \Sigma T_\alpha(x)_{ii}$ where T_r denotes the usual trace. Then the spherical functions for $G|_H$ are the functions φ_α defined by $\varphi_\alpha = \chi_\alpha * m_H$ for $\alpha \in \hat{G}$. The properties of φ_α are similar to the properties of $\psi_p P_m^\lambda$, $m = 0, 1, 2, \cdots$ of section 2 (the so called Gegenbauer polynomials) [cfr. 2, 3].

DEFINITION 3.1. Suppose (T, X) is a representation of G on a Banach space X. Then we call an operator $S \in \mathscr{B}(X)$ a G-operator if $ST(x) = T(x)S$ for all $x \in G$, where $\mathscr{B}(X)$ is the Banach algebra of bounded linear operators on X.

We now state and outline the proof of the main result of this section.

THEOREM 3.2. *Let* $T : L'_H(G) \to L'_H(G)$ *be a* G*-operator. Then* T *is a compact operator if and only if there exists a* $g \in L'_{HH}(G)$ *such that* $Tf = g * f$ *for each* $f \in L'_H(G)$.

Proof. Since finite linear combinations of the spherical functions φ_α are dense in $C_H(G)$ the same arguments of Theorem 2.1 show that T is compact.

Suppose T is a compact G-operator, then Theorem 9.3.6 of [3] implies the existence of $\mu \in M_{HH}(G)$ such that $Tf = \mu * f$ for all $f \in L'_H(G)$. Let V be a neighbourhood of the identity $e \in G$ such that $HVH = V$, then $m_H * U_V * m_H = U_V$ and $U_V \in L'_{HH}(G)$ with $\|U_V\| = 1$. We can show by a similar argument in Theorem 2.1 that there exists $\nu \in M_{HH}(G)$ such that

$\mu * U_V \to \nu$ in the weak*-topology and so for

$$f \in C_{HH}(G) \ , \ \int_G f(x) \, d(\mu * U_V) = \int_G f(x) \, d\nu(x).$$

It is routine to show that

$$\int_G f(x) \, d\nu(x) = \int_G (f * U_V)(y) \, d\mu(y).$$

Now $f * U_V \to f$ in the norm topology of $C_{HH}(G)$ and so $|f(x) - (f * U_V)(x)| \to 0$ for each $x \in G$ and $|(f * U_V)(x)| \leq \| f * U_V \|_\infty \leq \| f \|_\infty \| U_V \|_1 = \| f \|_\infty$. Hence by the Lebesgue dominated convergence theorem

$$\int_G (f * U_V)(x) \, d\mu(y) \to \int_G f(y) \, d\mu(y)$$

hence

$$\int_G f(x) \, d\nu(x) = \int_G f(y) \, d\mu(y)$$

and so $\nu = \mu$ a.e. with $\mu * U_V \to \mu$ in the weak*-topology. Again $\{\mu * U_V\}$ forms a bounded net in $L'_{HH}(G)$ so a subsequence of it converges to $g \in L'_{HH}(G)$ in the L'-norm. Hence $g = \mu$ and $Tf = g * f$ for all $f \in L'_H(G)$.

COROLLARY 3.3. *Let* $T : C_H(G) \to C'_H(G)$ *be a* G-*operator. Then* T *is compact if and only if there exists* $g \in L'_{HH}(G)$ *such that* $Tf = g * f$ *for all* $f \in C_H(G)$.

Proof. We proceed in the same way as in Theorem 3.2.

REFERENCES

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