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Some integral representation theorems for a space of quasi-continuous functions and its dual

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Analisi funzionale. — *Some integral representation theorems for a space of quasi-continuous functions and its dual.* Nota di WILLIAM D. L. APPLING, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Per $a < b$ e Q.C.$_0$ $[a, b]$ insieme di tutte le funzioni da $[a, b]$ nel piano complesso, quasi continue e nulle in a si ottengono rappresentazioni di integralí «destri» per gli elementi di Q.C.$_0$ $[a, b]$ e per gli elementi di (Q.C.$_0$ $[a, b]$)* insieme con i corrispondenti teoremi di unicità.

1. INTRODUCTION

Suppose $[a, b]$ is a number interval, Q.C.$_0$ $[a, b]$ is the set of all functions from $[a, b]$ into the plane, quasi-continuous on $[a, b]$ and having value 0 at a, and J_L and J_R are functions from $[a, b]$ into Q.C.$_0$ $[a, b]$ defined as follows:

$$J_L(y)(x) = \begin{cases} 0 & \text{if } a \le x \le y \\ 1 & \text{if } y < x \le b \end{cases}$$

$$J_R(y)(x) = \begin{cases} 0 & \text{if } y = a \\ 0 & \text{if } a \le x < y \\ 1 & \text{if } a < y \le x . \end{cases}$$

In a previous paper [1] the Author demonstrated a theorem that gives necessary and sufficient conditions that a real functional, defined on a set S of real-valued bounded set functions defined on a given field **F** of sets, have a certain kind of integral representation in terms of the real-valued bounded finitely additive functions defined on **F**. This characterization, which we forebear stating here in detail because it does not have a direct bearing on the matters of the present discussion, furnishes motivation for the theorems of this paper; the aferomentioned conditions involve an "integral representation" for the functions of S. We adopt this point of view in showing the following very explicit and elementary integral (see section 2 for the definition of integral) representation theorems for the elements of Q.C.$_0$ $[a, b]$ and (Q.C.$_0$ $[a, b]$)*, respectively:

THEOREM 3.1 (Section 3). *If for each y in $[a, b]$, each of* Q (y) *and* P (y) *is a function from $[a, b]$ into the plane, then the following two statements are equivalent:*

(*) Nella seduta del 13 dicembre 1975.

1) *If* f *is in* Q.C.$_0$ $[a, b]$, *then each of the integrals written immediately below exists and*

$$f = (R)\int_a^b f(x-)\, dP(x) + (R)\int_a^b [f(x) - f(x-)]\, Q(x),$$

and

2) *If* $a < y \leq b$, *then* $Q(y) = J_R(y) - J_L(y)$, *and if* $a \leq y \leq b$. *then* $P(y) = -J_L(y) + P(b)$.

THEOREM 4.1 (Section 4). *Suppose* T *is in* (Q.C.$_0$ $[a, b])^*$. *Then* $\{(x, T(-J_L(x))) : a \leq x \leq b\}$ *is in the set* B.V. $[a, b]$ *of all functions from* $[a, b]$, *into the plane having bounded variation on* $[a, b]$ *and* $\{(x, T(J_R(x) - J_L(x))) : a \leq x \leq b\}$ *is in the set* $S[a, b]$ *of all functions from* $[a, b]$ *into the plane such that*

$$\left\{\sum_D |f(q)| : [p, q] \text{ in } D \text{ a subdivision of } [a, b]\right\}$$

is bounded. Furthermore, if each of g *and is a function from* $[a, b]$ *into the plane, then the following two statements are equivalent*:

1) *If* f *is in* Q.C.$_0$ $[a, b]$, *then each of the integrals written immediately below exists and*

$$T(f) = (R)\int_a^b f(x-)\, dg(x) + (R)\int_a^b [f(x) - f(x-)]\, h(x)$$

and

2) *If* $a < y \leq b$, *then* $h(y) = T(J_R(y) - J_L(y))$, *and if* $a \leq y \leq b$, *then* $g(y) = -T(J_L(y)) + g(b)$.

2. PRELIMINARY THEOREMS AND DEFINITIONS

Suppose $a \leq r < s \leq b$. As usual, the statement that D is a subdivision of $[r, s]$ means that D is a finite collection of nonoverlapping intervals filling up $[r, s]$; and if H is a subdivision of $[r, s]$, then the statement that G is a refinement of H means that G is a subdivision of $[r, s]$, every member of which is a subset of some member of H.

We shall not state formal definitions of the integrals we shall discuss, but simply say that in this paper all integrals are limits, with respect to refinements of subdivisions, of the appropriate sums, either for absolute value in the case of integrals involving only functions from $[a, b]$ into the plane, or for absolute value supremum norm in the case of integrals involving functions from $[a, b]$ into the set of all functions from $[a, b]$ into the plane. As usual, the symbol (R) preceding the integral sign denotes "right hand" integral. Thus, if each of f and g is a function from $[a, b]$ into the plane and

Z is a function from $[a, b]$ into the set of all functions from $[a, b]$ into the plane, then

$$(R)\int_a^b f(x)\,dg(x),$$

$$(R)\int_a^b f(x)\,g(x),$$

$$(R)\int_a^b f(x)\,dZ(x)$$

and

$$(R)\int_a^b f(x)\,Z(x)$$

denote the respective limits with respect to refinements of subdivisions, of sums of the form, where $[p, q]$ is in E,

$$\sum_E f(q)\,[g(q)-g(p)],$$

$$\sum_E f(q)\,g(q),$$

$$\sum_E f(q)\,[Z(q)-Z(p)]$$

and

$$\sum_E f(q)\,Z(q),$$

the first two sums converging for absolute value, and the second two sums converging for absolute value supremum norm.

For f in $Q.C._0[a, b]$ we shall let f^- denote $\{(x, f(x-)) : a \leq x \leq b\}$; it is understood that $f(a-) = 0$.

We end this section with an elementary observation about J_L that we shall use in subsequent sections. Suppose $a \leq p < q \leq b$ and x is in $[a, b]$. Then

$$-J_L(q)(x) - -J_L(p)(x) = \begin{cases} 0 & \text{if } x \leq p \\ 1 & \text{if } p < x \leq q \\ 0 & \text{if } q < x. \end{cases}$$

3. A REPRESENTATION THEOREM FOR $Q.C._0[a, b]$

In this section we prove Theorem 3.1, as stated in the introduction.

Proof of Theorem 3.1. Let g denote f^-. Suppose $0 < c$. There is a subdivision D of $[a, b]$ such that if $[p, q]$ is in D and $p < u \leq v \leq q$, then

$$|g(u) - g(v)| < c/2.$$

Suppose E is a refinement of D. Suppose $a \leq x \leq b$. If $a = x$, then

$$\left| g(x) - \left(\sum_{E} g(q) [- J_L(q) - - J_L(p)] \right)(x) \right| =$$

$$= \left| 0 - \sum_{E} g(q) [- J_L(q)(a) - - J_L(p)(a)] \right| =$$

$$= \left| 0 - \sum_{E} g(q) [- 1] [0 - 0] \right| = 0 .$$

Suppose $a < x \leq b$. There is exactly one $[r, s]$ in E such tha $r < x \leq s$, and exactly one $[t, u]$ in D such that $t \leq r < x \leq s \leq u$. Therefore

$$\left| g(x) - \left(\sum_{E} g(q) [- J_L(q) - - J_L(p)] \right)(x) \right| =$$

$$= \left| g(x) + \sum_{E} g(q) [J_L(q)(x) - J_L(p)(x)] \right| =$$

$$= | g(x) + g(s) [J_L(s)(x) - J_L(r)(x)] | = | g(x) - g(s) | < c/2 .$$

Therefore

$$\left\| g - \sum_{E} g(q) [- J_L(q) - - J_L(p)] \right\| \leq c/2 < c .$$

Therefore

$$g = (R) \int_{b}^{a} g(x) \, d [- J_L(x)].$$

Now, consider $f - g$. Suppose $0 < c$. There is a subdivision D of $[a, b]$ such that if $[p, q]$ is in D and $p < u < q$, then $| f(u) - f(u-) | < c/2$. Suppose E is a refinement of D and $a \leq x \leq b$. Then

$$\left| f(x) - f(x-) - \left(\sum_{E} [f(q) - f(q-)] [J_R(q) - J_L(q)] \right)(x) \right| =$$

$$= \left| f(x) - f(x-) - \left(\sum_{E} [f(q) - f(q-)] [J_R(q)(x) - J_L(q)(x)] \right) \right| .$$

Now, if for some $[p, q]$ in E, $x = q$, then

$$\left| f(x) - f(x-) - \sum_{E} [f(q) - f(q-)] [J_R(q)(x) - J_L(q)(x)] \right| =$$

$$= | f(x) - f(x-) - [f(x) - f(x-)] [1] | = 0.$$

If for no $[p, q]$ in E, $x = q$, then either $x = a$, in which case

$$\left| f(x) - f(x-) - \sum_{E} [f(q) - f(q-)] [J_R(q)(x) - J_L(q)(x)] \right| =$$

$$= \left| 0 - \sum_{E} [f(q) - f(q-)] [0] \right| = 0,$$

or $a < x$, in which case there is exactly one $[r, s]$ in E such that $r < x < s$, and exactly one $[t, u]$ in D such that $t \leq r < x < s \leq u$, so that

$$\left| f(x) - f(x-) - \sum_{E} [f(q) - f(q-)] [J_R(q)(x) - J_L(q)(x)] \right| =$$

$$= \left| f(x) - f(x-) - \sum_{E} [f(q) - f(q-)] [0] \right| = | f(x) - f(x-) | < c/2.$$

Therefore

$$\left\| f - f^{-} - \sum_{E} [f(q) - f(q-)] [J_R(q) - J_L(q)] \right\| \leq c/2 < c.$$

Therefore

$$f - f^{-} = (R) \int_a^b [f(x) - f(x-)] [J_R(x) - J_L(x)].$$

We now easily see that it is an immediate consequence of the above representations for f^{-} and $f - f^{-}$ that 2) implies 1).

Now suppose 1) is true. Suppose $a < y \leq b$. Let $W = J_R(y) - J_L(y)$. Clearly

$$W = (R) \int_a^b W(x-)\, dP(x) + (R) \int_a^b [W(x) - W(x-)]\, Q(x) =$$

$$= (R) \int_a^b O dP(x) + (R) \int_a^b W(x)\, Q(x) = Q(y).$$

Now suppose that $a \leq y \leq b$. Then

$$J_L(y) = (R) \int_a^b J_L(y)(x-)\, dP(x) + (R) \int_a^b [J_L(y)(x) - J_L(y)(x-)] Q(x) =$$

$$= (R) \int_b^a J_L(y)(x)\, dP(x) + (R) \int_a^b 0 Q(x) = (R) \int_a^b J_L(y)(x)\, dP(x) +$$

$$+ 0 = (R) \int_a^b J_L(y)(x)\, dP(x).$$

Now, if $y = b$, then

$$J_L(b) = (R)\int_a^b J_L(b)(x)\,dP(x) = 0 = P(b) - P(b),$$

so that, in this case,

$$P(y) = -J_L(y) + P(b).$$

If $y < b$, then for each subdivision D of $[a, b]$ such that for some t, $[y, t]$ is in D,

$$\sum_D J_L(y)(q)[P(q) - P(p)] = \sum_{D'} J_L(y)(q)[P(q) - P(p)] =$$

$$= \sum_{D'} [1][P(q) - P(p)] = P(b) - P(y),$$

where $D' = \{[p, q] : [p, q] \text{ in } D, y < q\}$,
so that

$$J_L(y) = P(b) - P(y)$$

so that

$$P(y) = -J_L(y) + P(b).$$

Therefore 1) implies 2).
Therefore 1) and 2) are equivalent.

4. THE DUAL REPRESENTATION THEOREM

We begin this section by stating a thorem, the proof of which the reader can easily verify:

THEOREM 4.0. *Suppose b is in* B.V. $[a, b]$ *and h is in* S $[a, b]$. *Then, if w is in* $Q.C._0 [a, b]$, *then each of the integrals*

$$(R)\int_a^b w(x)\,dg(x) \qquad \text{and} \qquad (R)\int_a^b w(x)\,h(x)$$

exists. Furthermore, the function, T, *from* $Q.C._0 [a, b]$ *given, for each f in* $Q.C._0 [a, b]$, *by*

$$T(f) = (R)\int_a^b f(x-)\,dg(x) + (R)\int_a^b [f(x) - f(x-)]\,h(x),$$

is in $(Q.C._0 [a, b])^*$.

We now prove Theorem 4.1, as stated in the introduction.

Proof of Theorem 4.1. First, suppose D is a subdivision of $[a, b]$.

For each $[p, q]$ in D there is $w[p, q]$ in $\mathbf{R}\times\mathbf{R}$ such that

$|w[p, q]| = 1$ and $w[p, q]\, T(J_L(p) - J_L(q)) = |T(J_L(p) - J_L(q))|$,

so that

$$\sum_D |T(-J_L(q)) - T(-J_L(p))| = \left|\sum_D |T(J_L(p) - J_L(q))|\right| =$$

$$= \left|\sum_D w[p, q]\, T(J_L(p) - J_L(q))\right| = \left|T\left(\sum_D w[p, q](J_L(p) - J_L(q))\right)\right| \leq$$

$$\leq |T| \left\| \sum_D w[p, q](J_L(p) - J_L(q)) \right\| ,$$

where $|T|$ denotes then norm of T.

Now, if $a \leq x \leq b$, then there is at most one $[r, s]$ in D such that $r \leq x \leq s$ and $[J_L(r) - J_L(s)](x) = 1$, so that

$$\left|\left\{\sum_D w[p, q][J_L(p) - J_L(q)]\right\}(x)\right| \leq |w[r, s]|\, |1| = 1 ,$$

which implies that

$$\left\|\sum_D w[p, q][J_L(p) - J_L(q)]\right\| \leq 1 ,$$

so that

$$\sum_D |T(-J_L(q) - T(-J_L(p))| \leq |T| ,$$

so that

$$\{(x, T(-J_L(x))) : a \leq x \leq b\}$$

is in B.V. $[a, b]$.

Similarly, for each $[p, q]$ in D there is $v[p, q]$ in $\mathbf{R}\times\mathbf{R}$ such that

$|v[p, q]| = 1$ and $v[p, q]\, T(J_R(q) - J_L(q)) = |T(J_R(q) - J_L(q))|$,

so that

$$\sum_D |T(J_R(q) - J_L(q))| = \left|\sum_D v[p, q]\, T(J_R(q) - J_L(q))\right| =$$

$$= \left|T\left(\sum_D v[p, q][J_R(q) - J_L(q)]\right)\right| \leq |T| \left\|\sum_D v[p, q][J_R(q) - J_L(q)]\right\| .$$

Now, suppose $a \leq x \leq b$. If for no q such that $[p, q]$ is in D, $x = q$, then

$$\left|\left\{\sum_D v[p, q][J_R(q) - J_L(q)]\right\}(x)\right| = 0 .$$

If for some s such that $[r, s]$ is in D, $x = s$, then

$$\left|\left\{\sum_D v[p, q][J_R(q) - J_L(q)]\right\}(x)\right| = |v[r, s]\, 1| = 1 .$$

Therefore

$$\left\| \sum_{D} v[p,q]\,[J_R(q) - J_L(q)] \right\| \leq 1\,,$$

so that

$$\sum_{D} |T(J_R(q) - J_L(q))| \leq |T|\,,$$

so that

$$\{(x\,,T(J_R(x) - J_L(x))) : a \leq x \leq b\}$$

is in S $[a\,,b]$.

Now, if D is a subdivision of $[a\,,b]$, then

$$\left| T(f^{-}) - \sum_{D} f(q-)\,[T(-J_L(q)) - T(-J_L(p))] \right| +$$

$$+ \left| T(f - f^{-}) - \sum_{D} [f(q) - f(q-)]\,[T(J_R(q) - J_L(q))] \right| =$$

$$= \left| T(f^{-} - \sum_{D} f(q-)\,[-J_L(q) - - J_L(p)]) \right| +$$

$$+ \left| T(f - f^{-} - \sum_{D} [f(q) - f(q-)]\,[J_R(q) - J_L(q)] \right| \leq$$

$$\leq |T| \left\| f^{-} - \sum_{D} f(q-)\,[-J_L(q) - - J_L(p)] \right\| +$$

$$+ |T| \left\| f - f^{-} - \sum_{D} [f(q) - f(q-)]\,[J_R(q) - J_L(q)] \right\| ;$$

which, by Theorem 3.1, clearly implies that

$$T(f^{-}) = (R)\int_a^b f(x-)\, dT(-J_L(x))$$

and

$$T(f - f^{-}) = (R)\int_a^b [f(x) - f(x-)]\, T(J_R(x) - J_L(x))$$

We now easily see that the above discussion implies that 2) implies 1).

Now, suppose 1) is true. Suppose $a < y \leq b$. Let $W = J_R(y) - J_L(y)$. Then

$$T(W) = (R)\int_a^b W(x-)\, dg(x) + (R)\int_a^b [W(x) - W(x-)]\, h(x) =$$

$$(R)\int_a^b 0\, dg(x) + (R)\int_a^b W(x)\, h(x) = 0 + h(y)\,.$$

Now suppose $a \leq y \leq b$. Then

$$T(J_L(y)) \leq (R)\int_a^b J_L(y)(x-)\,dg(x) +$$

$$+(R)\int_a^b [J_L(y)(x) - J_L(y)(x-)]\,h(x) = (R)\int_a^b J_L(y)(x)\,dg(x) + 0.$$

If $y = b$, then

$$0 = T(J_L(b)) = (R)\int_a^b J_L(b)(x)\,dg(x) = 0 = g(b) - g(b),$$

so that, in this case

$$g(b) - g(y) = T(J_L(y)).$$

If $y < b$, then for each subdivision D of $[a, b]$ such that for some t, $[y, t]$ is in D,

$$\sum_D J_L(y)(q)\,[g(q) - g(p)] = \sum_{D'} J_L(y)(q)\,[g(q) - g(p)] =$$

$$= \sum_{D'} [1]\,[g(q) - g(p)] = g(b) - g(y), \quad \text{where } D' = \{[p, q] \text{ in } D : y < q\},$$

so that in this case,

$$T(J_L(y)) = g(b) - g(y).$$

Therefore 1) implies 2).
Therefore 1) and 2) are equivalent.

REFERENCES

[1] W. D. L. APPLING (1975) – *A Representation Characterization Theorem*, « Proc. Amer. Math. Soc. », 50, 317–321.