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On the behaviour of the solutions of a system of differential equations with set functions as unknowns. Nota I

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Equazioni funzionali. — *On the behaviour of the solutions of a system of differential equations with set functions as unknowns.* Nota I di ADOLF HAIMOVICI, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota I, ed in una Nota II che uscirà nel prossimo fascicolo, si studia il comportamento delle soluzioni del sistema (1), dove φ è una funzione vettoriale additiva d'insieme, con parte singolare ν. La soluzione banale di (1) si dice stabile se (6) implica (7) in un dominio Σ. Con questa definizione, e introducendo due funzioni esponenziali generalizzate (10_1), (10_2), (11_1), (11_2), si trovano condizioni sufficienti affinchè un sistema lineare (18), od anche non lineare, ammetta soluzioni stabili.

1. INTRODUCTION

Some years ago, I have proved existence and uniqueness theorems [1, 2] [(1)] for equations of the form:

$$\frac{d\varphi_i}{d\mu}(x) = F_i(x, \varphi(P_x)), \qquad (i = 1, 2, \cdots, m), \tag{1}$$

where

a) $x \in \Omega \subset R^k$ is a point in the domain;

b) μ is a weighted Lebsgue measure in R^k;

c) P is a mapping $\Omega \to \mathbf{P}(\Omega)$ continuous in the sense that, given $\varepsilon > 0$, it exists δ such that the inequality $\rho(x, x') < \delta$, leads to $\mu(P_x \Delta P_x) < \varepsilon$, where with Δ we have denoted the symmetric difference of the two sets P_x and $P_{x'}$;

d) φ_i are additive set functions, defined on $\mathbf{P}(\Omega)$;

e) $d\varphi_i/d\mu$ is the derivative of φ_i with respect to μ in the point x;

f) F_i are functions continuous in $\Omega \times R^m$, lipschitzian with respect to the last m variables.

Given a null-measure set H in R^k, and m singular measures with support on H [(2)], the system (1) is equivalent to the integral equations system

$$\varphi_i(P) = \nu_i(P \cap H) + \int_P F_i(y, \varphi(P_y))\, d\mu_y, \tag{2}$$

which for P_x leads to

$$\varphi_i(P_x) = \nu_i(P_x \cap H) + \int_{P_x} F_i(y, \varphi(P_y))\, d\mu_y. \tag{3}$$

(*) Nella seduta del 13 dicembre 1975.

(1) Numbers in brackets refer to the Bibliography given at the end of this « Nota I ».

(2) For questions regarding measure theory, see, for instance [5, 6, 7].

Putting

$$\varphi_i(\mathrm{P}_x) = u_i(x) ,$$

$$\nu_i(\mathrm{P}_x \cap \mathrm{H}) = \nu_i(x) ,$$

from the above system one gets the Volterra integral equations system

$$u_i(x) = v_i(x) + \int_{\mathrm{P}_x} \mathrm{F}_i(y , u(y))\, d\mu_y , \tag{5}$$

which can be solved by successive approximations, and gives $u_i(x) = \varphi_i(\mathrm{P}_x)$. Then, by (2), we obtain $\varphi_i(\mathrm{P})$.

Our aim is to discuss stability problems for some classes of systems (1), with the following definition of stabiity:

DEFINITION. *If the system* (1) *admits the trivial solution, we shall call this solution stable, if, given* $\varepsilon > 0$, *one can find* $\delta > 0$ *such that every solution of* (1) *with singular measure* $\nu_i(x)$ *and*:

$$\| \nu(\mathrm{P}_x \cap \mathrm{H}) \| < \delta \qquad \textit{for every} \quad x \in \Omega , \tag{6}$$

satisfies the inequality

$$\| \varphi(\mathrm{P}_x) \| < \delta \qquad \textit{for every} \quad x \in \Omega , \tag{7}$$

where we have denoted

$$\| \varphi \|^2 = \sum_1^m \varphi_i^2 . \tag{8}$$

The class of systems we shall consider has the form

$$\frac{d\varphi}{d\mu}(x) = \mathrm{A}\varphi(\mathrm{P}_x) + \mathrm{G}(x , \varphi(\mathrm{P}_x)) , \tag{9}$$

where:

a) φ is an m-dimensional vector function;

b) A is a constant matrix;

c) G is an n-dimensional vector function, satisfying certain restrictive conditions, which will be stated in the following.

2. PRELIMINARIES

I. For the following discussions we shall give here some results we have obtained in [4] concerning generalized exponential functions.

Let us consider the two equations

$$\text{(A)} \qquad \omega(x, \lambda, a) = a + \lambda \int_{P_x} \omega(\xi, \lambda, a)\, d\mu_\xi ,$$

$$\text{(B)} \qquad w(x, y, \lambda, a) = \chi_{P_x}(y) + \lambda \int_{P_x} w(\xi, y, \lambda, a)\, d\mu_\xi ,$$

where a is real, λ a complex parameter, and χ_P the characteristic function of the set P.

Suppose on the mapping P, besides the hypothesis c), the following ones

i) the mapping P is monotonic, i.e. if $y \in P_x$ then $P_y \subset P_x$;

ii) for every $\rho_0 > 0$, and every $x \in \Omega$, there exists a partition of P_x in a finite number of disjoint measurable parts $P_{(i)}$ of equal measure, of diameter at most ρ_0, satisfying the conditions:

α) if $\xi_i \in P_{(i)}$ and $P_{(j)} \subset P_{\xi_i}, j \neq i$, then, for every $\xi_j \in P_{(j)}$ we have $P_{(i)} \cap P_{(j)} = \varnothing$,

β) Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of partitions of P_x, $\{P_{(i)}^{(n)}\}_{i=1}^{N(n)}$ the set of parts of Δ_n, $\xi_i^{(n)} \in P_{(i)}^{(n)}$ and ρ_{0n} the norm of the partition Δ_n chosen so that $\lim_{n\to\infty} N(n) = \infty$ and $\lim_{n\to\infty} \rho_{0n} = 0$; denote by $X_a^{(n)}$ the set of points $\xi_i^{(n)}$ for which $P(\xi_i^{(n)})$ is covered by a parts $P_j^{(n)}$, by $N_a^{(n)}$ the number of points of $X_a^{(n)}$ and by p the number of values of a for which $X_a^{(n)} \neq \varnothing$; then

$$pN_a^{(n)} \leq N^{(n)} + \sigma\left(\frac{1}{N^{(n)}}\right) .$$

If hypotheses i), ii) α are satisfied, the solutions ω and w of (A) and (B) satisfy the following relations, closely analogous to that satisfied by the exponential function:

A) the functions ω and w can be represented as:

$$(10_1) \qquad \omega(x, \lambda, a) = a \sum_{0}^{\infty} \lambda^n \mu_n(x) ,$$

$$(10_2) \qquad w(x, y, \lambda, a) = a \sum_{0}^{\infty} \lambda^n \mu_n(x, y) ,$$

where

$$(11_1) \qquad \mu_0(x) = 1 \quad , \quad \mu_1(x) = \mu(P_x) \quad , \quad \mu(x) = \int_{P_x} \mu_{i-1}(\xi)\, d\mu_\xi ,$$

$$(11_2) \qquad \mu_0(x, y) = \chi_{P_x}(y) \quad , \quad \mu_i(x, y) = \int_{P_x} \mu_{i-1}(\xi, y)\, d\mu_\xi , \ (\mu_1(x, y) = \mu(P_{x,y})),$$

where $P_{x,y}$ is the set of points t such that $P_y \subset P_t \subset P_x$;

B) the functions ω and w are increasing functions of λ, for λ real and $\lambda \geq 0$;

C) the functions ω and w are increasing functions of x for $\lambda \geq 0$, in the sense that

$$P_x \subset P_{x'} \Rightarrow \begin{cases} \omega(x, \lambda, a) \leq \omega(x', \lambda, a) \\ w(x, y, \lambda, a) \leq w(x', y, \lambda, a); \end{cases}$$

D) if $y \notin P$, then $\mu_i(x, y) = 0$ and $w(x, y, \lambda, a) = 0$;

E) the function $w(x, y, \lambda, a)$ is decreasing in y, for $\lambda \geq 0$ in an analogous sense as in C;

F) (12_1) $w(x, y, \lambda, a) w(y, z, \lambda, 1) \leq w(x, z, \lambda, a)$, for $\lambda \geq 0$,

(12_2) $w(x, y, \lambda, a)\, \omega(y, \lambda, 1) \leq \omega(x, \lambda, a)$ for $\lambda \geq 0$.

If, besides hypotheses i, and ii α), the hypothesis ii β) is also satisfied, then the following inequalities hold:

$$(13_1) \qquad \omega(x, \lambda, a) \leq a \exp \lambda\mu(P_x), \qquad \lambda \geq 0,$$

$$(13_2) \qquad w(x, y, \lambda, a) \leq a \exp [\lambda\mu(x, y)], \qquad \lambda \geq 0.$$

Finally, we remark that one can consider $\omega(x, \lambda, a)$ and $w(x, y, \lambda, a)$ as set functions

$$\omega(P_x, \lambda, a) \quad \text{and} \quad w(P_x, y, \lambda, a);$$

and define their prolungation by

$$\omega(P, \lambda, a) = a + \lambda \int_P \omega(P_y, \lambda, a)\, d\mu_y,$$

$$w(P, y, \lambda, a) = a\chi_P(y) + \lambda \int_P w(P_z, y, \lambda, a)\, d\mu_z.$$

These functions satisfy the differential equations

$$(14_1) \qquad \frac{d\omega}{d\mu}(x, \lambda, a) = \lambda\omega(x, \lambda, a),$$

$$(14_2) \qquad \frac{dw}{d\mu}(x, y, \lambda, a) = \lambda w(x, y, \lambda, a).$$

If we add to this last two equations the conditions that the singular parts of ω, and resp w, have support on the origin and on y respectively, and have there the value a, then these equations are equivalent to (A) and (B).

Besides the functions ω and w, in [4] were also defined the functions

$$(15) \qquad w_t(x, y, \lambda) = \sum_{r=t-1}^{\infty} \binom{r}{t-1} \lambda^{r-\lambda-1} \mu_r(x, y),$$

which satisfy the equation

$$\frac{dw_t}{d\mu}(x, y, \lambda) = \lambda w_t(x, y, \lambda) + w_{t-1}(x, y, \lambda). \tag{15'}$$

It is obvious that

$$w(x, y, \lambda, a) = w_1(x, y, \lambda, a).$$

In what follows, we shall denote

$$\omega(x, \lambda) = \omega(x, \lambda, 1)$$

$$w(x, y, \lambda) = w(x, y, \lambda, 1).$$

Starting with (10_2) and (15), if λ is complex, we define the real functions

$$\begin{aligned} V_s(x, y, \lambda) &= \tfrac{1}{2}\{w_s(x, y, \lambda) + w_s(x, y, \bar{\lambda})\} = \sum_{k=s}^{\infty} \binom{k}{s} \operatorname{Re} \lambda^{k-s} \mu_k(x, y), \\ W_s(x, y, \lambda) &= \frac{1}{2i}\{w_s(x, y, \lambda) - w_s(x, y, \lambda)\} = \sum_{k=s}^{\infty} \binom{k}{s} \operatorname{Im} \lambda^{k-s} \mu_k(x, y). \end{aligned} \tag{17}$$

3. The derivative of an integral

The following theorem needs a new hypothesis on the mapping $P : \Omega \to \mathbf{P}(\Omega)$:

γ) For each additive set function Φ, defined on $\mathbf{P}(\Omega)$ and for each point $x_0 \in \Omega$, were Φ is derivable, there exists a family of sets $\{A_s\}$ containing x_0, such that:

γ_1) for each s, there exist p points $x_s^i(x_0)$ $(i = 1, 2, \cdots, p)$ and p constants α_i depending only on x_0 and not on s nor on Φ, satisfying the conditions

$$\Phi(A_s) = \sum_{i=1}^{p} \alpha_i \Phi(P_{x_s^i}), \qquad \lim_{s \to \infty} x_s^i = x_0.$$

γ_2) $\lim_{s \to \infty} \delta(A_s) = 0$.

From the continuity of Φ in x_0, one easily sees that it follows:

$$\sum_1^p \alpha_i = 0. \tag{18}$$

Theorem. *If $f(P, y)$ is a continuous function defined on $\mathbf{P}(\Omega) \times \Omega$, additive with respect to* P, *and $\varphi(P)$ is another additive set function satisfying*

$$\varphi(P_x) = \int_{P_x} f(P_x, y) \, d\mu_y,$$

then

$$\frac{d\varphi}{d\mu}(x_0) = f(P_{x_0}, x_0) + \int_{P_{x_0}} \frac{\partial f}{\partial \mu}(x_0, y)\, d\mu_y .$$

Proof. By hypothesis γ), for each x_0 there exists a family $\{A_s\}$ of sets, containing x_0, with $\delta(A_s) = 0$, for which

$$\frac{d\varphi}{d\mu}(x_0) = \lim_{s\to\infty} \frac{\varphi(A_s)}{\mu(A_s)},$$

and such that

$$\varphi(A_s) = \sum_{i=1}^{p} \alpha_i \varphi(P_{x_s^i}) \quad , \quad \sum_{i=1}^{p} \alpha_i = 0 , \lim_{s\to\infty} x_s^i = x_0 ;$$

with this we have

$$\varphi(A_s) = \sum_{i=1}^{p} \alpha_i \int_{P_{x_s^i}} f(P_{x_s^i}, y)\, d\mu_y = \sum_{i=1}^{p} \alpha_i \int_{P_{x_s^i}} f(P_{x_0}, y)\, d\mu_y +$$

$$+ \sum_{i=1}^{p} \alpha_i \int_{P_{x_s^i}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y =$$

$$= \int_{A_s} f(P_{x_0}, y)\, d\mu_y + \sum_{i=1}^{p} \alpha_i \int_{P_{x_0}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y +$$

$$+ \sum_{i=1}^{p} \alpha_i \int_{P_{x_s^i} \setminus P_{x_0}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y - \sum_{i=1}^{p} \alpha_i \int_{P_{x_0} \setminus P_{x_s^i}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y =$$

$$= \int_{A_s} f(P_{x_0}, y)\, d\mu_y + \int_{P_{x_0}} f(A_s, y)\, d\mu_y + \sum_{i=1}^{p} \alpha_i \int_{P_{x_s^i} \setminus P_{x_0}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y -$$

$$- \sum_{i=1}^{p} \alpha_i \int_{P_{x_0} \setminus P_{x_s^i}} [f(P_{x_s^i}, y) - f(P_{x_0}, y)]\, d\mu_y .$$

Divide now by $\mu(A_s)$ and let $s \to \infty$: the stated formula follows.

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