Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.1, p. 21–26.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_1_21_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Analisi funzionale. — *Stably-solvable operators in Banach spaces.*
Nota di MASSIMO FURI (*), MARIO MARTELLI (*) e ALFONSO VIGNOLI (**), presentata (***) dal Socio G. SANSONE.

RIASSUNTO. — Siano E , F due spazi di Banach, f e $g : E \to F$ due funzioni continue. In questa Nota preliminare si danno dei criteri per stabilire l'esistenza di soluzioni dell'equazione $f(x) = g(x)$. In un lavoro di prossima pubblicazione verranno esposte le relative dimostrazioni insieme ad alcune applicazioni alle equazioni differenziali ordinarie e alle derivate parziali.

Let E , F be Banach spaces and let $f, g : E \to F$ be two continuous maps. In this preliminary Note a method for proving the existence of solutions of the equation $f(x) = g(x)$ is given. The proofs, further results and some applications to ordinary and partial differential equations will appear in a forthcoming paper.

We recall that a continuous map $h : E \to F$ is quasibounded (see A. Granas [1]) if it maps bounded sets into bounded sets and

$$|h| = \limsup_{\|x\| \to \infty} \frac{\|h(x)\|}{\|x\|} < +\infty$$

The number $|h|$ is called the quasinorm of h.

DEFINITION 1. Let $f : E \to F$ be continuous. The map f is said to be *stably-solvable* if the equation $f(x) = h(x)$ has a solution provided that $h : E \to F$ is continuous, compact (i.e. it sends bounded sets into compact sets) and $|h| = 0$.

Note that any stably-solvable map $f : E \to F$ is onto. The converse is not true.

THEOREM 1. *The two following statements are equivalent*

i) *the identity* $I : E \to E$ *is stably-solvable*;

ii) *Schauder's fixed point theorem holds.*

THEOREM 2. *Let* $A : E \to F$ *be linear and continuous. Then* A *is stably-solvable if and only if it is onto.*

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Work performed under the auspices of the National Research Council of Italy (C.N.R.).
(***) Nella seduta del 10 gennaio 1976.

Let E , F , G be Banach spaces, $A : E \to F$, $L : E \to G$ be linear continuous and onto. Then the following conditions are equivalent

a) $\mathrm{KerA} + \mathrm{KerL} = E$;

b) A/KerL is onto;

c) L/KerA is onto;

d) the map $M : E \to F \times G$ defined by $Mx = (Ax , Lx)$ is onto.

THEOREM 3. *Let* $A : E \to F , L : E \to G$ *be linear continuous and onto. Let* $h : E \to F$ *be continuous compact and* $|h| = 0$. *Then the problem*

$$\begin{cases} Ax + h(x) = 0 \\ Lx = y \end{cases}$$

has a solution for any $y \in G$ *provided that* A *and* L *satisfy one of the conditions a)-d).*

Example 1. *Cauchy's Problem.* Let $f : [0 , 1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and such that $\|f(t , x)\| \leq M$ for every $(t , x) \in [0 , 1] \times \mathbf{R}^n$. It is known that Cauchy's Problem

$$\begin{cases} x' + f(t , x) = 0 \\ x(0) = a , \quad a \in R^n \end{cases}$$

has a solution (Peano's Theorem).

Write the above problem in the following form

$$\begin{cases} Dx + h(x) = 0 \\ Lx = a \end{cases}$$

where $D , h : C_n^1 [0 , 1] \to C_n [0 , 1]$, $L : C_n^1 [0 , 1] \to \mathbf{R}^n$ are defined by $Dx(t) = x'(t)$, $h(x)(t) = f(t , x(t))$, $Lx = x(0)$. It is easy to see that all of the conditions of Theorem 3 are verified. Thus Cauchy's Problem has at least one solution.

We recall that the map $g : E \to G$ is *asymptotically linear* (see M. A. Krasnosel'skij [2]) if

$$\lim_{\|x\| \to \infty} \frac{g(x) - g'_\infty(x)}{\|x\|} = 0$$

where $g'_\infty : E \to G$ is linear and continuous. The map g'_∞ is called the *asymptotic derivative* of g.

THEOREM 4. *Let* $A : E \to F , h : E \to F$ *be continuous. Assume that* A *is linear and onto, h is compact with* $|h| = 0$. *Let* $g : E \to G$ *be continuous, asymptotically linear and such that* $g - g'_\infty$ *is compact (this is the case if, for example,* G *is finite dimensional). If* g'_∞/KerA *is onto then the problem*

$$\begin{cases} Ax + h(x) = 0 \\ g(x) = y \end{cases}$$

has a solution for any $y \in G$.

Example 2. Consider the boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0 \\ x(0) + \left(\int_0^1 x^2(t)\, dt \right)^{1/3} = 1 \\ x(1) + x''(0) = 0 \end{cases}$$

where $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous and such that $|f(t, x, y)| \leq M$ for any $(t, x, y) \in [0, 1] \times \mathbf{R}^2$.

Put $E = C^2[0, 1]$, $F = C[0, 1]$, $G = \mathbf{R}^2$ and define $A, h: E \to F$, $g: E \to G$ by

$$Ax(t) = x''(t), \ h(x)(t) = f(t, x(t), x'(t)),$$

$$g(x) = \left(x(0) + \left(\int_0^1 x^2(t)\, dt \right)^{1/3}, x(1) + x''(0) \right).$$

One can show that all of the conditions of Theorem 4 are verified. Hence the boundary value problem has at least one solution.

In the above theorems we have assumed the perturbation h to be " small at infinity ", i.e. $|h| = 0$. The following result is a tool to deal with problems when h does not satisfy this condition.

CONTINUATION PRINCIPLE. *Let $f: E \to F$ be stably-solvable and $h: E \times [0, 1] \to F$ be continuous compact and such that $h(\cdot, 0) = 0$. Assume that the image $f(S)$ of the set $S = \{x \in E : f(x) + h(x, t) = 0, t \in [0, 1]\}$ is bounded. Then the equation $f(x) + h(x, 1) = 0$ has a solution.*

Remarks 1) The classic result of H. Schaefer [3] can be derived from the Continuation Principle by putting $f = I: E \to E$.

2) The assumption "$f(S)$ is bounded " can be replaced by the weaker condition " the connected component of $f(S)$ which contains the origin is bounded ".

3) Let $A: E \to F$, $L: E \to G$ be bounded and linear. Let $h: E \times [0, 1] \to F$ be continuous compact and such that $h(\cdot, 0) = 0$. Consider the problem

$$\begin{cases} Ax + h(x, 1) = 0 \\ Lx = y \end{cases}$$

where y is a given point of G. Assume that $A(S)$ is bounded, where $S = \{x \in E : Ax + h(x, t) = 0, Lx = y, t \in [0, 1]\}$. Using the Continuation Principle one can show that the above problem has a solution provided that A is onto and L/KerA is onto.

Example 3. Let $f: [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and such that $\|f(t, x)\| \leq A + B\|x\|$. Let $y \in R^n$. Consider the problem

$$\begin{cases} x' + sf(t, x) = 0 \\ x(0) = y. \end{cases}$$

Put $E = C_n^1[0, 1]$, $F = C_n[0, 1]$, $G = \mathbf{R}^n$ and define $A: E \to F$, $h: E \times [0, 1] \to F$, $L: E \to \mathbf{G}$ by $Ax(t) = x'(t)$, $h(x, s)(t) = sf(t, x(t))$, $Lx = x(0)$.

Using the Gronwall Lemma one can show that all of the conditions of Remark 3 are verified. Hence the problem

$$\begin{cases} x' + f(t, x) = 0 \\ x(0) = y \end{cases}$$

has a solution.

As a consequence of the Continuation Principle we have the following

THEOREM 5. *Let* $A: E \to F$, $L: E \to G$ *be linear continuous and onto. Let* $h: E \to F$ *be continuous and compact. Assume that* L/KerA *is onto. Then there exists* $\varepsilon > 0$ *such that the problem*

$$\begin{cases} Ax + \lambda h(x) = 0 \\ Lx = y \end{cases}$$

has a solution for any $\lambda \in (-\varepsilon, \varepsilon)$.

Example 4. Consider the equation

$$(e)\ x' + B(t)x = \lambda f(t, x)$$

where $B(t)$ is an $n \times n$ continuous matrix, periodic of period T, $f: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $f(t, x) = f(t + T, x)$ for any $t \in R$. Assume that the only T-periodic solution of the homogeneous system $x' + B(t)x = 0$ is the trivial one.

We want to show that there exists $\varepsilon > 0$ such that the equation (e) has a T-periodic solution for any $\lambda \in (-\varepsilon, \varepsilon)$ (see G. Villari [4]).

Let $E = C_n^1[0, T]$, $F = C_n[0, T]$. Define $A, h: E \to F$, $L: E \to R^n$ by $Ax(t) = x'(t) + B(t)x(t)$, $h(x)(t) = f(t, x(t))$, $Lx = x(0) - x(T)$.

The equation (e) has a T-periodic solution if and only if the problem

$$\begin{cases} Ax = \lambda h(x) \\ Lx = 0 \end{cases}$$

has a solution. It is easy to see that all of the conditions of Theorem 5 are verified. Hence the above problem has a solution for λ sufficiently small.

In Theorem 3 we proved that under suitable conditions the problem

$$(p) \begin{cases} Ax + h(x) = 0 \\ Lx = y \end{cases}$$

has a solution for any $y \in G$. We may ask if the set of solutions of (p) depends continuously on y. A positive answer to this question can be given in the case when L/KerA is an isomorphism. More precisely the following theorem holds.

THEOREM 6. *Let* $A : E \to F$, $L : E \to G$ *be linear and bounded. Assume that* A *is onto and* L/KerA *is an isomorphism. Let* $h : E \to F$ *be continuous compact with* $|h| = 0$. *Then the multi-valued map* $S : F \times G \multimap E$ *which assignes to every* (y, z) *the set of solutions of the problem*

$$\begin{cases} Ax + h(x) = y \\ Lx = z \end{cases}$$

is upper semicontinuous (with compact values).

Note that S is continuous in the case when S is single-valued.

Let $A : E \to F$ be an isomorphism and $K : E \to F$ be linear and compact. It is known that if $Ax + Kx = 0$ implies that $x = 0$ then the equation $Ax + Kx = y$ has a solution for any $y \in F$ (Fredholm alternative). The following theorem can be regarded as a generalization of the above classic result.

THEOREM 7. *Let* $A : E \to F$, $K : E \to F$, $L : E \to G$ *be linear and continuous. Assume that* A *is onto*, L/KerA *is onto and* K *is compact. If*

$$\begin{cases} Ax + Kx = 0 \\ Lx = 0 \end{cases}$$

implies that $Ax = 0$ *then the problem*

$$\begin{cases} Ax + Kx + h(x) = 0 \\ Lx = y \end{cases}$$

has a solution for any $y \in G$, *provided that* $h : E \to F$ *is continuous compact with* $|h| = 0$.

Remark. Let A, K be as in Theorem 7. If $Ax + Kx = 0$ implies that $Ax = 0$ then $A + K$ is stably-solvable (hence it is onto).

We may ask if the results given in this preliminary note hold true in the framework of Fréchet spaces. The answer is positive, even though some definitions and theorems need to be suitably modified. In particular a linear continuous map $A : E \to F$, where E, F are Fréchet spaces, is stably-solvable if and only if it is onto.

A result analogous to Theorem 3 can be obtained. As an example one can show that the following problem has a solution

$$\begin{cases} x' + f(t, x) = 0 \\ x(0) = a \qquad a \in \mathbf{R}^n \end{cases}$$

with $f : [0, +\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ continuous and such that $|f(t, x)| \leq g(t)$, where $g : [0, +\infty) \to \mathbf{R}$ is locally bounded.

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