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Sufficient Conditions for Nonoscillation of n-th Order Nonlinear Differential Equations

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Equazioni differenziali ordinarie. — *Sufficient Conditions for Nonoscillation of n-th Order Nonlinear Differential Equations* (*). Nota di LU–SAN CHEN, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni sufficienti perché tutte le soluzioni di una classe di equazioni differenziali nonlineari siano nonoscillatorie.

1. INTRODUCTION

In the paper [1], Graef and Spikes have given sufficient conditions for every solution of the nonlinear differential equation

$$[l(t)\,x'(t)]' + p(t)\,g(x(t), x'(t)) = h(t; x(t), x'(t))$$

to be nonoscillatory. Those results are obtained in terms of integral conditions on $h(t; x(t), x'(t))$ and $p(t)$. The purpose of the present paper is to give sufficient conditions for nonoscillation of n-th order nonlinear differential equations of the form

$$(*)\qquad [l(t)\,x^{(n-1)}(t)]' + p(t)\,g(x(t), x'(t), \cdots, x^{(n-1)}(t)) = h(t; x(t), \cdots, x^{(n-1)}(t)), \qquad (n \geq 2)$$

which contains a damping term involving the $(n-1)$-th derivative of the unknown function, where

(i) $l : [t_0, \infty) \to (0, \infty)$ is continuous, $t_0 \geq 0$,

(ii) $p : [t_0, \infty) \to (0, \infty)$ is continuous,

(iii) $g : R^n \to R = (-\infty, \infty)$ is continuous and satisfies

$y_1\, g(y_2, \cdots, y_n) > 0$ for $y_1 (\neq 0)$ and for any $t (\geq t_0)$,

(iv) $h : [t_0, \infty) \times R^n \to R$ is continuous.

In the sequel, all functions considered will be assumed continuous on their domains, and the existence of solutions of (*), which are valid for all large t, will be assumed without further mention. A solution of (*) is a function $x(t) \in C^n([t_x, \infty), R)$, which satisfies (*) on $[t_x, \infty)$ ($t_x \geq t_0$ (t_0 fixed) and depending on the particular solution $x(t)$). Denote by U the family of all such solutions of (*). A function $x(t) \in U$ is said to be "nonoscillatory" if there exists $t_1 (\geq t_x)$ such that $x(t) \neq 0$ for $t \geq t_1$ and it is said to be bounded if $|x(t)| \leq k$ for every $t \in [t_x, \infty)$, where k is a positive constant.

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2. Sufficient conditions for nonoscillation.

To prove the main results we need the following Lemma.

LEMMA (Kiguradze [2], [3]). *Let u be a positive ν-times continuously differentiable function on an interval* $[t_0, \infty)$. *If $u^{(\nu)}$ is of constant sign and is not identically zero for all large t, then there exist a* $T (\geq t_0)$ *and an integer* $l, 0 \leq l \leq \nu$ *with* $\nu + l$ *odd if* $u^{(\nu)} \leq 0$, $\nu + l$ *even if* $u^{(\nu)} \geq 0$ *and such that for every* $t \geq T$,

$$l > 0 \quad \textit{implies} \quad u^{(k)} > 0 \qquad (k = 0, 1, \cdots, l-1)$$

and

$$l \leq \nu - 1 \quad \textit{implies} \quad (-1)^{l+k} u^{(k)}(t) > 0, \quad (k = l, l+1, \cdots, \nu - 1).$$

We make use of the following conditions:

(c_1) for every $x(t) \in C^n [t_0, \infty)$ there exists a constant M such that

$$g(x(t), x'(t), \cdots, x^{(n-1)}(t)) \leq M \qquad \text{for all large } t;$$

(c_2) for every $x(t) \in C^n [t_0, \infty)$ there exists a constant L such that

$$L \leq g(x(t), x'(t), \cdots, x^{(n-1)}(t))$$

for all large t;

(c_3) for every $x(t) \in C^n [t_0, \infty)$ there exists a real-valued function $h_1(t)$ on $[t_0, \infty)$ such that

$$h_1(t) \leq h(t; x(t), x'(t), \cdots, x^{(n-1)}(t))$$

for all large t;

(c_4) for every $x(t) \in C^n [t_0, \infty)$ there exists a real-valued function $h_2(t)$ on $[t_0, \infty)$ such that

$$h(t; x(t), x'(t), \cdots, x^{(n-1)}(t)) \leq h_2(t)$$

for all large t.

THEOREM 1. *Suppose that conditions* (c_1) *and* (c_3) *are satisfied. Moreover, assume that*

$$\text{(1)} \qquad \int_{t_0}^{\infty} [h_1(s) - Mp(s)]\, ds = +\infty .$$

Then every $x(t) \in U$ is nonoscillatory.

Proof. We will write equation (*) as the system

$$(**) \qquad \begin{cases} x^{(n-1)} = \dfrac{y}{l(t)}, \\ y' = h(t; x(t), \cdots, x^{(n-1)}(t)) - p(t) g(x(t), \cdots, x^{(n-1)}(t)) . \end{cases}$$

Suppose there is a solution $(x(t), y(t))$ of (**).
Integrating the second equation in (**) from t_0 to t, we have

$$y(t) = y(t_0) + \int_{t_0}^{t} h(s; x(s), \cdots, x^{(n-1)}(s))\, ds -$$

$$- \int_{t_0}^{t} p(s) g(x(s), \cdots, x^{(n-1)}(s))\, ds .$$

Then, from (c_1), (c_3) and (1) we have

$$y(t) \geq y(t_0) + \int_{t_0}^{t} [h_1(s) - \mathrm{M}p(s)]\, ds \to +\infty \qquad \text{as} \quad t \to \infty .$$

Hence, there exists $t_1 \geq t_0$ such that $y(t) \geq 0$ for $t \geq t_1$, which, in view of the first equation in (**), implies $x^{(n-1)}(t) \geq 0$ for $t \geq t_1$. Hence, from the Lemma, there exist a $\mathrm{T}\,(\geq t_1)$ and an integer $l\,(0 \leq l \leq n-1)$ with $n-1+l$ even, such that for every $t \geq \mathrm{T}$

$$x^{(k)}(t) > 0 , \qquad k = 0, 1, \cdots, l-1,$$

$$(-1)^{l+k} x^{(k)}(t) > 0 , \qquad k = l, l+1, \cdots, n-2.$$

From this we conclude that $x'(t) > 0$ for every $t \geq \mathrm{T}$ and so $x(t)$ is non-oscillatory. Here no assumption is made about the sign of M.

The proof of following theorem follows from a procedure quite similar to the proof of Theorem 1. The details may be omitted.

THEOREM 2. *Suppose that conditions (c_2) and (c_4) are satisfied. Moreover, assume that*

$$\int_{t_0}^{\infty} [h_2(s) - \mathrm{L}p(s)]\, ds = -\infty .$$

Then every $x(t) \in \mathrm{U}$ is nonoscillatory, where no assumption is made about the sign of L.

THEOREM 3. *For the equation (*), subject to the condition (c_3), suppose that $g(y_1, y_2, \cdots, y_n)$ is boudend from above whenever the first variable is bounded. Moreover, assume that*

$$\int_{t_0}^{\infty} [h_1(s) - \lambda_1 p(s)]\, ds = +\infty \qquad \textit{for every} \quad \lambda_1 .$$

Then every bounded $x(t) \in \mathrm{U}$ is nonoscillatory.

Proof. Suppose that $x(t) \in \mathrm{U}$ is bounded with the property $0 < x(t) < k$ for every $t \geq t_0$. Then there exists a constant λ_1 such that $0 < g(x(t), \cdots, x^{(n-1)}(t)) < \lambda_1$ for every $t \geq t_0$. Hence we obtain

$$y(t) \geq y(t_0) + \int_{t_0}^{t} [h_1(s) - \lambda_1 p(s)] \, ds \rightarrow +\infty \qquad \text{as} \quad t \rightarrow \infty,$$

and the desired conclusion follows as in Theorem 1. Thus $x(t)$ is nonoscillatory. A similar proof holds in the case $-k < x(t) < 0$ ($k =$ a positive constant) and our theorem is established.

THEOREM 4. *For the equation* (*), *subject to the condition* (c_4), *suppose that* $g(y_1, y_2, \cdots, y_n)$ *is bounded from below whenever the first variable is bounded. Moreover, assume that*

$$\int_{t_0}^{\infty} [h_2(s) - \lambda_2 p(s)] \, ds = -\infty \qquad \textit{for every} \quad \lambda_2.$$

Then every bounded $x(t) \in \mathrm{U}$ *is nonoscillatory.*

Proof. As $g(y_1, y_2, \cdots, y_n)$ is bounded from below, i.e. $\lambda_2 < g(y_1, y_2, \cdots, y_n)$, the theorem follows immediately from Theorem 3.

THEOREM 5. *Suppose the condition* (c_3), *in addition assume that there exists a constant* μ *satisfying*

$$|g(y_1, y_2, \cdots, y_n)| \leq \mu \qquad \textit{for every} \quad t \geq t_0$$

and

$$\int_{t_0}^{\infty} [h_1(s) - \mu |p(s)|] \, ds = +\infty.$$

Then every $x(t) \in \mathrm{U}$ *is nonoscillatory.*

Proof. The proof is obvious.

THEOREM 6. *In Theorem 5, replace* (c_3) *by* (c_4) *and, moreover, suppose that*

$$\int_{t_0}^{\infty} [h_2(s) + \mu |p(s)|] \, ds = -\infty.$$

Then every $x(t) \in \mathrm{U}$ *is nonoscillatory.*

Proof. This follows immediately from Theorem 5.

The proofs of the following theorems are similar to the proofs of the above ones and will be omitted.

THEOREM 7. *Assume that the condition* (c_3) *is satisfied and that* $g(y_1, y_2, \cdots, y_n)$ *is bounded whenever the first variable is bounded. Moreover, suppose*

$$\int_{t_0}^{\infty} [h_1(s) - \tau |p(s)|] \, ds = +\infty \qquad \textit{for every} \quad \tau > 0.$$

Then every bounded $x(t) \in U$ *is nonoscillatory.*

THEOREM 8. *In Theorem 7, replace* (c_3) *by* (c_4) *and moreover, assume*

$$\int_{t_0}^{\infty} [h_2(s) + \tau |p(s)|] \, ds = -\infty \qquad \textit{for every} \quad \tau > 0.$$

Then every bounded $x(t) \in U$ *is nonoscillatory.*

Remark 1. The particular case $n = 2$ is due to Graef and Spikes [1].

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