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Right alternative alternator ideal algebras

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Algebra. — *Right alternative alternator ideal algebras* (*). Nota di GIULIA MARIA PIACENTINI CATTANEO, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Si studiano algebre R sopra un campo F alternative a destra, soddisfacenti un'identità della forma $[a, (a, a, b)] = \gamma (a, a, [a, b])$, per $\forall a, b \in R$ e qualche γ in F, tali inoltre che il sottogruppo additivo generato dagli alternatori sia un ideale. Si dimostra che, se queste algebre sono prime e dotate di unità 1 e di un idempotente $e \neq 0, \neq 1$, allora (salvo poche eccezioni qui specificate) esse sono alternative. Si suppone sempre che la caratteristica del campo F sia diversa da 2 e da 3.

INTRODUCTION

Let R be a nonassociative algebra over a field F. An associator (a, b, c) is defined as usual as $(a, b, c) = (ab) c - a (bc)$ and a commutator $[a, b]$ as $[a, b] = ab - ba$. An alternator is an associator of the form (a, a, b), (a, b, a) or (b, a, a). We shall make the assumption on the characteristic of F to be prime to 6, so that scalar factors 1/2 and 1/3 are admissible.

An alternator ideal algebra is an algebra in which the additive subgroup generated by all alternators is an ideal, with the condition that there also be a formula for the absorption. For more details on alternator ideal algebras we refer to [3].

In [5] it has been shown that, if R is a right alternative algebra, then R is an alternator ideal algebra if, and only if, $[R, M(R, R, R)] \subseteq M(R, R, R)$, where $M(a, b, c) = (a, b, c) + (b, a, c)$. In this paper we study right alternative algebras R which for all $a, b, c \in R$, satisfy an identity of type

$$(1) \qquad [a, (b, b, c)] = \alpha_1 M([a, b], b, c) + \alpha_2 M([a, b], c, b) + \alpha_3 M([a, c], b, b) + \alpha_4 M([b, c], a, b) + \alpha_5 M([b, c], b, a)$$

with $\alpha_i \in F$.

In [3] it was shown that, if R is a right alternative alternator ideal algebra with an identity element 1, then R satisfies (1) if and only if, for some $\gamma \in F$, $[a, (a, a, b)] = \gamma (a, a, [a, b])$. We are thus studying right alternative alternator ideal algebras such that $[a, (a, a, b)] = \gamma (a, a, [a, b])$ for some

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γ in F. In [3] it was shown that, with two possible exceptions, if R is a prime, right alternative algebra satisfying an identity of the form (1), then, if R has an identity element 1 and an idempotent e, $e \neq 0$, $\neq 1$ such that $(e, e, R) = 0$, R is alternative.

In this paper we drop the requirement $(e, e, R) = 0$ and we prove the following:

THEOREM. *If* R *is a right alternative algebra with an identity element* 1 *and an idempotent* $e \neq 0$, $\neq 1$ *and if* R *satisfies* (1), *then* (e, e, R) *is a trivial ideal of* R, *contained in the center of* $R_1 + R_0$, *apart from some exceptions specified later on.*

COROLLARY. *If* R *is a prime right alternative algebra with an identity element* 1 *and an idempotent* $e \neq 0$, $\neq 1$ *and if* R *satisfies an identity of the form* (1), *then, with some possible exceptions specified later on,* R *is alternative.*

NOTATION

We shall use both juxtaposition and "$\cdot$" to indicate multiplication. In expressions where both appear, the product indicated by juxtaposition is to be taken first: thus $(a, b, c) = ab \cdot c - a \cdot bc$.

We shall often write expressions like (R, R, R) (or $[R, R]$) meaning by this the vector space over F generated by all (a, b, c) for all $a, b, c \in R$ (or by all $[a, b]$ for all $a, b \in R$).

If I is an ideal of an algebra R and $I^2 = 0$, we call I a *trivial* ideal. R is *semiprime* if it has no nonzero trivial ideals. R is *prime* if, whenever I and J are ideals such that $IJ = 0$, then $I = 0$ or $J = 0$.

The *right nucleus* of R is the set $\{a \in R / (R, R, a) = 0\}$. The *left* and *middle nucleus* are defined analogously. The *nucleus* of R is the set $\{a \in R / (R, R, a) = (R, a, R) = (a, R, R) = 0\}$. The *center* of R is the set $\{a \in R / [a, R] = (R, R, a) = (R, a, R) = (a, R, R) = 0\}$.

An *alternative algebra* R is an algebra such that $(a, b, b) = (b, a, b) = (b, b, a)$ for all $a, b \in R$.

A *right alternative algebra* R is an algebra such that $(a, b, b) = 0$ for all $a, b \in R$. The following identities hold in any right alternative algebra of characteristic different from 2 (see [4]):

$$(2) \qquad (a, b, bc) = (a, b, c)\, b, \qquad \text{for all } a, b, c \in R,$$

$$(3) \quad \bar{A}(a, b, c, d) = (ab, c, d) + (a, b, [c, d]) - a(b, c, d) - (a, c, d)\, b = 0, \qquad \text{for all } a, b, c, d \in R.$$

If R is a right alternative algebra over a field of characteristic different from 2, with an idempotent e, let $R = R_1 + R_{1/2} + R_0$ be the Albert decomposition of R relative to e, where $ex_1 = x_1 e = x_1$, $ex_0 = x_0 e = 0$ and $e_{1/2} + x_{1/2}\, e = x_{1/2}$ (see [1]). It is easy to see that $R_1 R_0 = R_0 R_1 = 0$. If we consider the algebra $R^{(+)}$ which is the same vector space as R and has product $x \circ y$ defined in terms of the product xy of R by $2x \circ y = xy + yx$, $R^{(+)}$ is a *Jordan algebra* (see Theorem 2 of [2]). The following inclusions then hold (see [2]): $R_{1/2} \circ R_1 \subseteq R_{1/2}$, $R_{1/2} \circ R_0 \subseteq R_{1/2}$ and $R_{1/2} \circ R_{1/2} \subseteq R_1 + R_0$. Also, we have $(e, e, R) = (e, e, R_{1/2}) \subseteq R_1 + R_0$. If we furthermore set $R_{10} = \{y \in R / y = ex_{1/2} + (e, e, [e, x_{1/2}])$ for some $x_{1/2} \in R_{1/2}\}$ and $R_{01} = \{y \in R / y = x_{1/2} e + (e, [e, x_{1/2}], e)$ for some $x_{1/2} \in R_{1/2}\}$ then $R = R_1 + R_{10} + R_{01} + R_0$ and, once we observe that $eR_{1/2} \cdot e = 0$, it is a matter of simple computation to show that $R_{10}, R_{01} \subseteq R_{1/2}$ and that $R_{10} e \subseteq R_1$, $eR_{01} \subseteq R_0$.

We shall use the subscript notation to specify a particular component of a product we intend to examine. Thus, for example, $(x_{01} y_{01})_0$ means the summand of the product $x_{01} y_{01}$ which lies in R_0.

When an algebra R satisfies (1) with coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ we will say that R is of type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

PRELIMINARY RESULTS

From now on, R will always represent a right alternative algebra with an identity element 1 and an idempotent $e \neq 0, \neq 1$, which satisfies an identity of the form (1). We furthermore suppose the characteristic of F to be prime to 6. Since we are interested in the case in which (e, e, R) is not identically 0, the following lemma will hold.

LEMMA 1. $\alpha_1 + \alpha_2 + \alpha_3 = 0$ *and* $\alpha_3 + \alpha_4 + \alpha_5 = 0$.

Proof. From (1) we derive

$$0 = [x, (e, e, e)] = (\alpha_1 + \alpha_2 + \alpha_3)(e, [x, e], e),$$

$$0 = [e, (e, e, x)] = (\alpha_3 + \alpha_4 + \alpha_5)(e, [e, x], e).$$

By our assumption on (e, e, R), the lemma is proved.

LEMMA 2. $R_1 R_1 \subseteq R_1$.

Proof. By $(x_1, y_1, e) = -(x_1, e, y_1) = 0$ it follows that $x_1 y_1 = a_1 + a_{01} - ea_{01}$. To prove that $R_1 R_1 \subseteq R_1$ it is therefore sufficient to show that $(x_1 y_1)_{01} = 0$.

From (1), $[e, M(e, x_1, y_1)] = [e, (e, x_1, y_1)] = [e, x_1 y_1] =$ by Lemma 1 $\alpha_3 (e, e, [x_1, y_1])$ which tells that $[e, x_1 y_1]$ is an element of $R_1 + R_0$ and hence $(x_1 y_1)_{01} = 0$.

COROLLARY 1. $R_0 R_0 \subseteq R_0$.

Proof. Since R has an identity element 1, the idempotent $e' = 1 - e$ satisfies $e' \neq 0, \neq 1$ and $(e', e', R) \neq 0$. The decomposition of R with respect to e' is exactly the same as the one with respect to e, except that the subscripts are interchanged. Thus the corollary is proved by "reversing subscripts".

LEMMA 3. a) $R_{10} R_1 \subseteq R_{10} e$, b) $R_1 R_{10} \subseteq eR_{10}$, c) $R_{10} R_0 \subseteq eR_{10}$, d) $R_0 R_{10} \subseteq R_{10} e$.

Proof. From $(x_{10}, y_1, e) = -(x_{10}, e, y_1)$ and Lemma 2 we derive $R_{10} R_1 \subseteq R_1$. From $(y_1, x_{10}, e) = -(y_1, e, x_{10})$ and $(e, y_1, x_{10}) = -(e, x_{10}, y_1)$, it follows $R_1 R_{10} \subseteq eR_{10}$. Since $R_{10} \circ R_1 \subseteq R_{1/2}$, then $R_{10} R_1 \subseteq R_{10} e$.

From $(x_{10}, y_0, e) = -(x_{10}, e, y_0)$, $(y_0, x_{10}, e) = -(y_0, e, x_{10})$ and $x_{10} \circ y_0 \in R_{1/2}$, we have $y_0 x_{10} = a_{10} e + b_{01} - eb_{01}$, $x_{10} y_0 = -a_{10} e + a_{10} + eb_{01}$. By addition of the following two equalities

$$[e, (y_0, e, x_{10}) + (e, y_0, x_{10})] = -[e, y_0 x_{10}] = (\alpha_3 + \alpha_5) M([e, x_{10}], y_0, e) + \\ + (\alpha_3 + \alpha_4) M([e, x_{10}], e, y_0) + (\alpha_4 + \alpha_5) M([y_0, x_{10}], e, e)$$

and

$$0 = [y_0, (e, e, x_{10})] = \alpha_3 M([y_0, x_{10}], e, e) + \alpha_4 M([e, x_{10}], y_0, e) + \\ + \alpha_5 M([e, x_{10}], e, y_0),$$

and by Lemma 1, $R_0 R_{10} \subseteq R_1 + R_0$, which yields $R_0 R_{10} \subseteq R_{10} e$ and $R_{10} R_0 \subseteq eR_{10}$.

COROLLARY 2. a') $R_{01} R_0 \subseteq eR_{01}$, b') $R_0 R_{01} \subseteq R_{01} e$, c') $R_{01} R_1 \subseteq R_{01} e$, d') $R_1 R_{01} \subseteq eR_{01}$.

Proof. It follows from Lemma 3 interchanging subscripts.

MAIN SECTION

We are now able to study the set $(e, e, R) = R_{10} e + eR_{01}$. We shall prove the results for $R_{10} e$; hence, reversing the subscripts, analogous results will hold for eR_{01}. The results contained in the following lemmas will allow us to say that (e, e, R) is a trivial ideal of R, contained in the center of $R_1 + R_0$. There are some exceptions to this statement.

LEMMA 4. *$R_{10} e$ is a trivial ideal of R_1, contained in the center of R_1.*

Proof. From Lemma 2 and 3 it easily follows that $M([e, x_{10}], e, y_1) = -x_{10} y_1$, $M([e, x_{10}], y_1, e) = -y_1 \cdot x_{10} e$ and $M([y_1, x_{10}], e, e) = (y_1 x_{10})_1 = -x_{10} y_1$. Adding the following two relations

$$0 = [e, (e, y_1, x_{10}) + (y_1, e, x_{10})] = -\alpha_4 x_{10} y_1 + \alpha_4 y_1 \cdot x_{10} e,$$

$$[y_1, (e, e, x_{10})] = \alpha_4 x_{10} y_1 - \alpha_4 y_1 \cdot x_{10} e$$

one gets

$$[R_1, (e, e, R_{10})] = [R_1, R_{10} e] = 0; \tag{4}$$

since $R_{10}e \cdot R_1 \subseteq R_{10}R_1 \subseteq R_{10}e$, the previous relation tells us that $R_{10}e$ is an ideal of R_1, which commutes with R_1.

We next show that $R_{10}e$ is contained in the nucleus of R_1. By $\bar{A}(x_{10}, e, y_1, z_1) = 0$ and Lemma 2 it follows that $(ex_{10}, y_1, z_1) = 0$. Again by $\bar{A}(e, x_{10}, y_1, z_1)$, whe have $(e, x_{10}, [y_1, z_1]) = (x_{10}, y_1, z_1)$, which yields $x_{10}e \cdot z_1 y_1 = (x_{10}e \cdot y_1) z_1$. By (4) and by the fact that $R_{10}e$ is an ideal of R_1 it follows $z_1 y_1 \cdot x_{10}e = z_1(y_1 \cdot x_{10}e)$ which tells us that $x_{10}e$ is in the right nucleus of R_1. The following equalities say that $x_{10}e$ is in the left nucleus of R_1, and hence in the nucleus of R_1:

$$(x_{10}e, y_1, z_1) = (x_{10}e \cdot y_1) z_1 - x_{10}e \cdot y_1 z_1 = z_1 y_1 \cdot x_{10}e - y_1 z_1 \cdot x_{10}e =$$
$$= z_1 y_1 \cdot x_{10}e - y_1(z_1 \cdot x_{10}e) = z_1 y_1 \cdot x_{10}e - (y_1 \cdot x_{10}e) z_1 = (z_1, y_1, x_{10}e) = 0.$$

To conclude the proof of the lemma we must now show that the R_1-ideal $R_{10}e$ is trivial. By (4) it is sufficient to show that $(x_{10}e)^2 = 0$. By $x_{10}^2 = (x_{10}e + ex_{10})x_{10} = x_{10}e \cdot x_{10} + ex_{10}^2$ and by the fact that $x_{10}^2 \in R_1$ from $(e, x_{10}, x_{10}) = 0$, $R_{1/2} \circ R_{1/2} \subseteq R_1 + R_0$ and Lemma 3, it follows $x_{10}e \cdot x_{10} = 0$. Then, on account of (2),

$$0 = x_{10}e \cdot x_{10} = (e, e, x_{10})x_{10} = (e, x_{10}e, x_{10}) = -(e, x_{10}, x_{10}e) =$$
$$- ex_{10} \cdot x_{10}e + x_{10} \cdot x_{10}e = (x_{10}e)^2.$$

COROLLARY 3. *(e, e, R) is a trivial ideal of $R_1 + R_0$, contained in the center of $R_1 + R_0$.*

Proof. By reversing subscripts in Lemma 4, eR_{01} is a trivial ideal of R_0, contained in the center of R_0. Since $(e, e, R) = R_{10}e + eR_{01}$ and since $R_1 R_0 = R_0 R_1 = 0$, the lemma si proved.

LEMMA 5. $R_{01} \cdot R_{10}e = 0$, *unless the algebra is of type* $(2\alpha, \alpha - 1, -3\alpha + 1, \alpha, 2\alpha - 1)$.

Proof. Right alternativity, Lemma 3 and Corollary 2 yield $R_{10}R_{01} \subseteq R_1 + R_0$ and $R_{01}R_{10} \subseteq R_1 + R_0$. If we set $x_{10}y_{01} = a_1 + a_0$, $y_{01}x_{10} = b_1 + b_0$, it is easy to check that the following relations hold:

$$x_{10}e \cdot y_{01} = a_0,\ ey_{01} \cdot x_{10} = b_1,\ M([e, x_{10}], e, y_{01}) = -3a_0 - x_{10} \cdot ey_{01},$$
$$M([e, x_{10}], y_{01}, e) = a_0 + x_{10} \cdot ey_{01} + b_1 - y_{01} \cdot x_{10}e,\ M([e, y_{01}], e, x_{10}) =$$
$$= -3b_1 - y_{01} \cdot x_{10}e,\ M([e, y_{01}], x_{10}, e) = b_1 + y_{01} \cdot x_{10}e + a_0 - x_{10} \cdot ey_{01}$$

with $x_{10} \cdot ey_{01} \in eR_{10}$ and $y_{01} \cdot x_{10}e \in R_{01}e$.

From (1) we derive the relations

$$[e, M(e, x_{10}, y_{01})] = \{-3\alpha_1 + \alpha_2 - \alpha_4\}a_0 + \{3\alpha_1 + 4(\alpha_2 - \alpha_4)\}b_1 +$$
$$+ \{-\alpha_1 + \alpha_2 + \alpha_4\}x_{10} \cdot ey_{01} + \{\alpha_1 - 2\alpha_4\}y_{01} \cdot x_{10}e,$$

$$[y_{01}, (e, e, x_{10})] = \{-3\alpha_1 - 4(\alpha_2 - \alpha_4)\}a_0 + \{3\alpha_1 - \alpha_2 + \alpha_4\}b_1 +$$
$$+ \{-\alpha_1 + 2\alpha_4\}x_{10} \cdot ey_{01} + \{\alpha_1 - \alpha_2 - \alpha_4\}y_{01} \cdot x_{10}e.$$

Since $[e, M(e, x_{10}, y_{01})] = -x_{10} \cdot ey_{01} + b_1$, and $[y_{01}, (e, e, x_{10})] = y_{01} \cdot x_{10} e - a_0$, the previous system becomes

$$0 = \{-3\alpha_1 + \alpha_2 - \alpha_4\} a_0 + \{3\alpha_1 + 4(\alpha_2 - \alpha_4) - 1\} b_1 + \{-\alpha_1 + \alpha_2 + \alpha_4 + 1\} x_{10} \cdot ey_{01} + \{\alpha_1 - 2\alpha_4\} y_{01} \cdot x_{10} e,$$

$$0 = \{-3\alpha_1 - 4(\alpha_2 - \alpha_4) + 1\} a_0 + \{3\alpha_1 - \alpha_2 + \alpha_4\} b_1 + \{-\alpha_1 + 2\alpha_4\} x_{10} \cdot ey_{01} + \{\alpha_1 - \alpha_2 - \alpha_4 - 1\} y_{01} \cdot x_{10} e.$$

Suppose $y_{01} \cdot x_{10} e \neq 0$. Then $\alpha_1 - 2\alpha_4 = 0$ and $\alpha_1 - \alpha_2 - \alpha_4 - 1 = 0$, and the algebra is of type $(2\alpha, \alpha - 1, -3\alpha + 1, \alpha, 2\alpha - 1)$, and it immediately follows that $a_0 = x_{10} e \cdot y_{01}$ must equal zero.

COROLLARY 4. $R_{10} \cdot eR_{01} = 0$, *unless the algebra is of type* $(2\alpha, \alpha - 1, -3\alpha + 1, \alpha, 2\alpha - 1)$.

Proof. It follows from the previous lemma interchanging subscripts.

LEMMA 6. $R_{10} \cdot R_{10} e = 0$. $R_{10} e \cdot R_{10} = 0$, *unless the algebra is of type* $(\alpha, \beta, -\alpha - \beta, 2\alpha + \beta - 1, -\alpha + 1)$.

Proof. By Lemma 4, $R_{10} e$ is a trivial ideal of R_1; then $0 = x_{10} e \cdot y_{10} e = x_{10} \cdot y_{10} e$ and the first statement is proved. By right alternativity, Lemma 3 and $R_{10} \circ R_{10} \subseteq R_1 + R_0$ we can write $x_{10} y_{10} = a_1 + a_{10} + a_{01} - ea_{01}$, $y_{10} x_{10} = b_1 - a_{10} - a_{01} + ea_{01}$. Also we have

$$x_{10} e \cdot y_{10} = -y_{10} e \cdot x_{10} = a_{10} - a_{10} e,$$

$$M([e, x_{10}], e, y_{10}) = -a_1 - a_{10} - a_{10} e - ea_{01},$$

$$M([e, x_{10}], y_{10}, e) = M([e, y_{10}], x_{10}, e) = a_1 + b_1,$$

$$M([x_{10}, y_{10}], e, e) = -2a_{10} e - 2ea_{01},$$

$$M([e, y_{10}], e, x_{10}) = -b_1 + a_{10} + a_{10} e + ea_{01}.$$

We now use (1) and the above equalities to get the following relations:

$$[e, M(e, x_{10}, y_{10})] = \{-\alpha_1 + \alpha_2 - \alpha_4\} a_1 + \{\alpha_1 + 2(\alpha_2 - \alpha_4)\} b_1 + \{-4\alpha_1 - 3\alpha_2 + \alpha_4\} (a_{10} e + ea_{01}) + \{-2\alpha_1 - \alpha_2 + \alpha_4\} a_{10},$$

$$0 = [e, M(x_{10}, y_{10}, e)] = \{3(\alpha_4 - \alpha_2)\} a_1 + \{3(\alpha_4 - \alpha_2)\} b_1,$$

$$[x_{10}, M(y_{10}, e, e)] = \{-\alpha_1 + \alpha_2 - \alpha_4\} a_1 + \{\alpha_1 + 2(\alpha_2 - \alpha_4)\} b_1 + \{-4\alpha_1 - 3\alpha_2 + \alpha_4\} (a_{10} e + ea_{01}) + \{-2\alpha_1 - \alpha_2 + \alpha_4\} a_{10}.$$

Since $[e, M(e, x_{10}, y_{10})] = -a_{10} + 2a_{10} e$, $[x_{10}, M(y_{10}, e, e)] = -a_{10} + a_{10} e$, it follows, subtracting the third equation from the first, $a_{10} e = 0$, which yields

$a_{10} = x_{10}\, e \cdot y_{10}$. The system becomes

$$\begin{aligned} 0 &= \{-\alpha_1 + \alpha_2 - \alpha_4\}\, a_1 + \{\alpha_1 + 2\,(\alpha_2 - \alpha_4)\}\, b_1 + \\ &\quad + \{-4\,\alpha_1 - 3\,\alpha_2 + \alpha_4\}\, ea_{01} + \{-2\,\alpha_1 - \alpha_2 + \alpha_4 + 1\}\, a_{10}\,, \\ 0 &= \{3\,(\alpha_4 - \alpha_2)\}\, a_1 + \{3\,(\alpha_4 - \alpha_2)\}\, b_1\,. \end{aligned}$$

From the last equation one sees that two cases may arise: either $\alpha_2 = \alpha_4$ or $a_1 + b_1 = 0$.

If $\alpha_2 = \alpha_4$ and $a_{10} \neq 0$, then $\alpha_1 = 1/2$ and the algebra is of type $(1/2\,, \alpha\,, -\alpha - 1/2\,, \alpha\,, 1/2)$. In these conditions, $a_1 = b_1$ and $ea_{01} = (x_{10}\, y_{10})_0 = 0$ unless $\alpha = -1$.

If $a_1 + b_1 = 0$ and $a_{10} \neq 0$, it follows $a_1 = -b_1 = 0$ and the algebra is of type $(\alpha\,, \beta\,, -\alpha - \beta\,, 2\,\alpha + \beta - 1\,, -\alpha + 1)$. If $\beta \neq -\alpha - 1/2$, then $ea_{01} = (x_{10}\, y_{10})_0 = 0$.

Algebras of type $(1/2\,, \alpha\,, -\alpha - 1/2\,, \alpha\,, 1/2)$ are clearly of type $(\alpha\,, \beta\,, -\alpha - \beta\,, 2\,\alpha + \beta - 1\,, -\alpha + 1)$.

COROLLARY 5. $R_{01} \cdot eR_{01} = 0$. $eR_{01} \cdot R_{01} = 0$, *unless the algebra is of type* $(\alpha\,, \beta\,, -\alpha - \beta\,, 2\,\alpha + \beta - 1\,, -\alpha + 1)$.

We are now able to obtain the following

THEOREM. *Let* R *be a right alternative algebra with identity element* 1 *and an idempotent* $e \neq 0\,, \neq 1$. *If* R *satisfies* (1), *then* $(e\,, e\,, R)$ *is a trivial ideal of* R, *contained in the center of* $R_1 + R_0$, *with only possible exceptions for algebras of type* $(2\,\alpha\,, \alpha - 1\,, -3\,\alpha + 1\,, \alpha\,, 2\,\alpha - 1)$ *or* $(\alpha\,, \beta\,, -\alpha - \beta\,, 2\,\alpha + \beta - 1\,, -\alpha + 1)$.

Proof. The proof follows from Corollary 3, Lemma 5, Lemma 6 and the corollaries to these lemmas, keeping also in mind that, by Lemma 3 and Corollary 2, $R_{10}\, e \cdot R_{01} \subseteq eR_{01} \subseteq (e\,, e\,, R)$ and $eR_{01} \cdot R_{10} \subseteq R_{10}\, e \subseteq (e\,, e\,, R)$.

COROLLARY 6. *Let* R *be a prime, right alternative algebra with an identity element* 1 *and an idempotent* $e \neq 0\,, \neq 1$. *If* R *satisfies* (1), *then* R *is alternative. The only possible exceptions are the ones listed in the previous theorem and* $(-1/2\,, 0\,, 1/2\,, 0\,, 1/2)$.

Proof. By the Theorem above and primeness, $(e\,, e\,, R)$ must be 0. The result then follows from Theorem 10 of [3]. Algebras of type $(1/2\,, -3/4\,, 1/4\,, 1/4\,, -1/2)$, which appear as exceptional cases in that theorem, are included in the algebras of type $(2\,\alpha\,, \alpha - 1\,, -3\,\alpha + 1\,, \alpha\,, 2\,\alpha - 1)$.

As a concluding remark we have that a quick glance at the exceptional cases of the Theorem allows us to say that, in algebras of types $(2\,\alpha\,, \alpha - 1\,, -3\,\alpha + 1\,, \alpha\,, 2\,\alpha - 1)$, if $\alpha \neq 1/2\,, (e\,, e\,, R)$ is a right ideal of R and a trivial ideal of $R_1 + R_0$, contained in the center of $R_1 + R_0$; in algebras of type $(\alpha\,, \beta\,, -\alpha - \beta\,, 2\,\alpha + \beta - 1\,, -\alpha + 1)$, if $\alpha \neq 1$ and $\beta \neq -1/2$, $(e\,, e\,, R)$ is a left ideal of R (and a trivial ideal of $R_1 + R_0$ contained in the center of $R_1 + R_0$).

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