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A regular 5-graph

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Teorie combinatorie. — *A regular 5-graph*. Nota di WILLEM MIELANTS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Mentre si conoscono vari eleganti t-grafi regolari per $t = 2$, per $t \geq 3$ se ne conoscono pochissimi; in particolare, per $t = 5$ ne era noto soltanto uno [1], collegato col sistema di Steiner L_{12} dovuto a Witt. Qui viene costruito e studiato un altro 5-grafo regolare, collegato col gruppo di Mathieu M_{24}.

1. Z_2-COHOMOLOGY OF k-SYSTEMS

We denote the set of all k-subsets of a set Ω by $\Omega^{|k|}$ and call a subset of $\Omega^{|k|}$ a k-system on Ω.

We define now the Z_2-coboundary operator $\delta : \text{Hom}(\Omega^{|k|}, Z_2) \rightarrow \text{Hom}(\Omega^{|k+1|}, Z_2)$ as follows. If $f \in \text{Hom}(\Omega^{|k|}, Z_2)$ and if $\alpha = \{x_1, x_2, \cdots x_{k+1}\} \in \Omega^{|k+1|}$ then

$$\delta f(\alpha) = \sum_{i=1}^{k+1} f(\hat{\alpha}_i) \qquad \text{with} \quad \hat{\alpha}_i = \alpha / \{x_i\}.$$

With each $f \in \text{Hom}(\Omega^{|k|}, Z_2)$ corresponds a k-system $\Delta(f)$ on Ω with $\alpha \in \Delta(f) \Longleftrightarrow f(\alpha) = 1$. Hence the Z_2-coboundary of a k-system $\Delta(f)$ on Ω is the set of all those $(k+1)$-subsets of Ω containing an odd number of blocks of $\Delta(f)$. A $(k+1)$-system on Ω with vanishing Z_2-coboundary (or a Z_2-cocycle) is called [1] a k-graph on Ω; and it is called a regular k-graph it it is also a k-design. The only known regular k-graphs with $k \geq 3$ [1] are: a regular 3-graph which is the design of non-planar 4-subsets of AG$(3, 2)$ on 8 points, and a double extension of the Petersen graph on 12 points, admitting the Mathieu group M_{11} in its, 3-transitive representation on 12 points as a group of automorphisms.

No models are known of regular 4-graphs and the design of 6-subsets which are no blocks of the Steiner system L_{12} of Witt is the only known example of a regular 5-graph.

If $f, g \in \text{Hom}(\Omega^{|k|}, Z_2)$ then the equivalence classes of k-systems on Ω defined by the equivalence relation $\Delta(f) \sim \Delta(g) \Longleftrightarrow \delta f \equiv \delta g$ are called the Seidel classes or switching classes of k-systems on the set Ω [1]. Interesting k-systems are of course transitive designs with k points on each block or the orbits or union of orbits of k-subsets of transitive permutation groups.

If G is a t-transitive permutation group on Ω which is not k-homogeneous and if $A \in \Omega^{|k|}$ then the orbit of $A : \{A^g \| g \in G\}$ is a $t - [|\Omega|, k, \lambda]$-design with $|G_A|^{-1} \cdot |G|$ blocks (G_A being the setwise stabilizer of G with respect to A). The only known 5-transitive permutation groups which are not trivial

(*) Nella seduta dell'8 maggio 1976.

(no symmetrical or alternating groups) are the Mathieu groups of degree 12 and 24. The Mathieu group M_{12} is the setwise stabilizer of M_{24} with respect to a special 12-subset. Now we shall make a study of the Z_2-coboundaries and the switching classes of the orbits of subsets of Ω_{24} by the Mathieu group M_{24}.

2. The Mathieu group M_{24}

The simple group PSL (3, 4) acting doubly transitive on the 21 points of the projective plane of order 4 : PG (2, 4) has a transitive extension: the simple Mathieu group of degree 22 acting triply transitive on $\{x_1^\infty\} \cup$ PG (2, 4). The orbit of $\{x_1^\infty\} \cup$ L where L is a line of PG (2, 4) by M_{22} is the Steiner system 3-(22, 6, 1) of Witt which we denote by L_{22} and which is an extension of the projective plane PG (2, 4). M_{22} has a transitive extension: the simple Mathieu group of degree 23: M_{23} acting quadruply transitive on $\{x_1^\infty\} \cup \{x_2^\infty\} \cup$ $\cup$ PG (2, 4) and the orbit of $\{x_1^\infty\} \cup \{x_2^\infty\} \cup$ L where L is a line of PG (2, 4) by M_{23} is the Steiner system 4-(23, 7, 1) of Witt which we denote by L_{23} and which is an extension of L_{22}. M_{23} has also a transitive extension: the simple Mathieu group of degree 24 acting quintuply transitive on $\{x_1^\infty\} \cup \{x_2^\infty\} \cup \{x_3^\infty\} \cup$ $\cup$ PG (2, 4) and the orbit of $\{x_1^\infty\} \cup \{x_2^\infty\} \cup \{x_3^\infty\} \cup$ L where L is a line of PG (2, 4) by M_{24} is the Steiner system 5-(24, 8, 1) of Witt which we denote by L_{24} and which is an extension of L_{23}.

M_{24} has no transitive extension [3]. M_{24} can also be defined as a permutation group on the 24 points of a projective line on the Galois field GF (23) obtained by adjoining to the group PSL (2, 23) the permutation α with $x^\alpha = 9x^3$ if x is a non-square of GF (23) and $x^\alpha = 1/9x^3$ is x is a square of GF (23) [2].

Consider a set Ω of 24 elements. By defining the sum of two subsets of Ω as their symmetrical difference, we obtain a 24-dimensional vector space over GF (2).

In this vector space M_{24} leaves invariant a 12-dimensional subspace C (called the perfect binary (24, 12) Golay-code). This perfect code contains 759 words of weight 8 and 2576 words of weight 12. The corresponding 8 and 12-subsets of Ω are called the special octads and the umbral dodecads. The special octads are the blocks of the Steiner system L_{24}.

Consider an umbral dodecad in L_{24} and denote 3 of its points by x_1^∞, x_2^∞ and x_3^∞, then the internal structure $(L_{24})_{\{x_1^\infty, x_2^\infty, x_3^\infty\}}$ is the projective plane of order 4 : PG (2, 4), and the 9 remaining points of the umbral dodecad are the 9 absolute points of a unitary polarity in PG (2, 4) [3].

The setwise stabilizer of M_{24} with respect to a special octad is the group 2^4. Alt (8) which is an extension of the alternating group of degree 8 by the elementary Abelian 2-group of order 16 acting regular on the 16 remaining points of L_{24}. The stabilizer of an arbitrary point is Alt (8) acting on the 15 remaining points equivalently as PSL (4, 2) on the points of PG (3, 2).

The setwise stabilizer of M_{24} with respect to an umbral dodecad is the Mathieu group M_{12}.

A n-subset of L_{24} with $n < 12$ is called a special n-ad if it contains or is contained in a special octad.

A n-subset of L_{24} with $5 < n < 12$ is called an umbral n-ad if it is contained in an umbral dodecad.

An n-subset of L_{24} with $7 < n < 12$ which is not a special n-ad or an umbral n-ad is called a transverse n-ad.

A non-umbral dodecad is called extra special if it contains three special octads, special if it contains exactly one special octad, penumbral if it contains all but one of the points of an umbral dodecad, and transverse in all other cases.

Sets of more than 12 points are described by the same adjectives as their complements.

We denote the set of special n-ads $(0 \leq n \leq 24)$ by S_n,

the set of umbral n-ads $(6 \leq n \leq 18)$ by U_n,

the set of transverse n-ads $(8 \leq n \leq 16)$ by T_n,

the set of extra special dodecads by S_{12}^{+},

and the set of penumbral dodecads by U_{12}^{-}.

Conway [2] has proved that those sets are exactly the 49 orbits of subsets of Ω_{24} by the Mathieu group M_{24}.

By proving the uniqueness of the extensions of PG $(2, 4)$, L_{22} and L_{23}, Lüneburg [3] has also found the following geometrical model of L_{24}. The points of L_{24} are the points of PG $(2, 4)$ together with 3 new points: x_1^{∞}, x_2^{∞} and x_3^{∞}.

The blocks of L_{24} are the lines of PG $(2, 4)$, the hyperovals of PG $(2, 4)$ (ovals together with their nucleus), the Baer subplanes of PG $(2, 4)$ (in this case the Fano-configurations), and the symmetric differences of pairs of lines in PG $(2, 4)$.

To define the incidences we have first to remark that the groups PSL $(3, 4)$ has exactly 3 orbits of hyperovals: $\Delta_1, \Delta_2, \Delta_3$ and exactly 3 orbits of Baer subplanes: U_1, U_2, U_3.

With each pair of orbits of hyperovals Δ_i, Δ_j corresponds exactly one orbit of Baer subplanes U_k with the property that for $\forall \alpha \in \Delta_i$, $\forall \beta \in \Delta_j$ and $\forall \gamma \in U_k \Rightarrow |\alpha \cap \gamma| \leq 3$ and $|\beta \cap \gamma| \leq 3$ $[i, j, k \in \{1, 2, 3\}\ i \neq j\ i \neq k\ j \neq k]$.

A point of PG $(2, 4)$ is incident with a block if it is incident with it in PG $(2, 4)$.

A point x_i^{∞} $(i = 1, 2, 3)$ is incident with each line of PG $(2, 4)$ with each hyperoval not belonging to the orbit Δ_i and with each Baer subplane of the orbit U_i.

With this incidence the points and blocks form the 5-$(24, 8, 1)$ Steiner-system of Witt.

3. THE REGULAR 5-GRAPH OF UMBRAL HEXADS OF M_{24}

An umbral hexad of M_{24} is a 6-subset of the Steiner system L_{24} not contained in a block. If we denote three of its points by x_1^∞, x_2^∞ and x_3^∞ then the three remaining points form a triangle in PG (2 , 4).

Since it admits M_{24} as a 5-transitive group of automorphisms it is a 5-design and since there are 16 possibilities for a sixth point of a block if 5 are given it is a 5-(24 , 6 , 16) design.

Now we prove that it is a regular 5-graph or that its Z_2-coboundary is zero or that each 7-subset of L_{24} contains an even number of umbral hexads.

Of course a special heptad contains no umbral hexads. Consider now an umbral heptad A. If we denote three of its points by $x_1^\infty, x_2^\infty, x_3^\infty$ then the remaining 4 points are not collinear in PG (2 , 4).

If they form a quadrangle ($B_1 B_2 B_3 B_4$) in PG (2 , 4) then there is a unique hyperoval incident with them (denote the orbit of hyperovals of PSL (3 , 4) to which it belongs by Δ_1).

Of course $A - \{B_i\}$ ($i = 1 , 2 , 3 , 4$) is always an umbral hexad. Also $A - \{x_2^\infty\}$ and $A - \{x_3^\infty\}$ are umbral hexads since the hyperoval incident with $B_1 B_2 B_3 B_4$ belongs not to the orbits Δ_2 and Δ_3. But $A - \{x_1^\infty\}$ is a special hexad since the 4 points $B_1 B_2 B_3 B_4$ belong to a hyperoval of the orbit Δ_1 and so $x_2^\infty, x_3^\infty, B_1 , B_2 , B_3 , B_4$ belong to a block of L_{24}.

If now $B_2 B_3$ and B_4 are collinear and B_1 is not incident with this line of course $A - \{x_i^\infty\}$ ($i = 1 , 2 , 3$) is always a umbral hexad. Also $A - \{B_i\}$ $i = 2 , 3 , 4$ are umbral hexads but $A - \{B_1\}$ is then a special hexad.

Hence each special heptad contains exactly zero umbral hexads and each umbral heptad contains exactly 6 umbral hexads. So each 7-subset of L_{24} contains an even number of umbral hexads and so the 5-(24 , 6 , 16) design of umbral hexads is a Z_2-cocycle or a regular 5-graph.

4. THE SEIDEL CLASSES OF ORBITS OF SUBSETS OF Ω_{24} BY M_{24}

In the same manner the Z_2-coboundaries of all the other orbits of subsets of Ω_{24} by M_{24} can be found.

$k = 6$ Two Seidel classes: S_6 and U_6.
The Z_2-coboundary of S_6 is $\Omega^{|7|}$ or the set of all 7 subsets.
The Z_2-coboundary of U_6 is zero (this is the regular 5-graph).

$k = 7$ One Seidel class.
Since $\Omega^{|7|}$ is itself a Z_2-coboundary, all orbits of 7-subsets are switching.
The Z_2-coboundary of S_7 and U_7 is the orbit of transverse octads (T_8).

$k = 8$ Three Seidel classes.
One 7-graph: T_8.
The Z_2-coboundary of the Steiner system L_{24} is S_9.
The Z_2-coboundary of U_8 is $T_9 \cup U_9$.

$k = 9$ One Seidel class.
The orbits of 9-subsets are all switching since S_9, T_9 and U_9 are all 8-graphs.

$k = 10$ Three Seidel classes
One 9-graph: T_{10}.
The Z_2-coboundary of S_{10} is S_{11}.
The Z_2-coboundary of U_{10} is $T_{11} \cup U_{11}$.

$k = 11$ Two Seidel classes
One 10-graph: S_{11}.
Since $T_{11} \cup U_{11}$ it itself a Z_2-coboundary T_{11} and U_{11} are switching, and have the orbit of penumbral dodecads as Z_2-coboundary.

$k = 12$ Four Seidel classes.
Two 11-graphs: T_{12} and U_{12}^-.
The Z_2-coboundaries of S_{12}^+, S_{12} and U_{12} are respectively S_{13}, T_{13} and U_{13}.

$k = 13$ One Seidel class.
S_{13}, T_{13} and U_{13} are all 12-graphs.

$k = 14$ Three Seidel classes.
One 13-graph: T_{14}.
The Z_2-coboundaries of S_{14} and U_{14} are respectively $S_{15} \cup T_{15}$ and U_{15}.

$k = 15$ Two Seidel classes.
One 14-graph: U_{15}.
Since $S_{15} \cup T_{15}$ is itself a Z_2-coboundary,
S_{15} and T_{15} are switching and have as Z_2-coboundary T_{16}.

$k = 16$ Three Seidel classes.
One 15-graph: T_{16}.
The Z_2-coboundaries of S_{16} and U_{16} are respectively S_{17} and U_{17}.

$k = 17$ One Seidel class.
S_{17} and U_{17} are both 16-graphs.

$k = 18$ Two Seidel classes.
One 17-graph: U_{18}.
The Z_2-coboundary of S_{18} is S_{19}.
For $k > 18$ all Z_2-cohomology is trivial.

Hence we have the following non-trivial 5-transitive Z_2-cocycles on 24 points: U_6, T_8, S_9, T_9, U_9, $S_9 \cup T_9$, $S_9 \cup U_9$, $T_9 \cup U_9$, T_{10}, S_{11}, $T_{11} \cup U_{11}$, T_{12}, U_{12}^-, $T_{12} \cup U_{12}^-$, S_{13}, T_{13}, U_{13}, $S_{13} \cup T_{13}$, $S_{13} \cup U_{13}$, $T_{13} \cup U_{13}$, T_{14}, U_{15}, $S_{15} \cup T_{15}$, T_{16}, S_{17}, U_{17} and U_{18}.

References

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